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Quantum scattering of giant magnons

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Abstract: We perform a first-principles semi-classical computation of the one-loop corrections to the dispersion relation and S-matrix of Giant Magnons in $AdS_5 \times S^5$ string theory. The results agree exactly with expectations based on the strong coupling expansion of the exact Asymptotic Bethe Ansatz equations. In particular we reproduce the Hernandez-Lopez term in the dressing phase.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Extended Supersymmetry.

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Contents

1. Introduction

Recent developments in the study of planar $\mathcal{N} = 4$ SUSY Yang-Mills (and the dual string theory on $AdS_5 \times S^5$) have culminated in a proposal for a set of Asymptotic Bethe Ansatz Equations (ABAE) [1−4, 30]. These equations determine the exact scaling dimensions Δ, of all operators in a limit where a conserved R-charge J becomes infinite, with the difference $\Delta - J$ and the 't Hooft coupling $\lambda = g^2_{\rm YM} N$ held fixed. The proposed equations hold for all values of λ, but for $\lambda >> 1$ their predictions can be compared to the results of semiclassical calculations in the worldsheet theory of the $AdS_5 \times S^5$ string. In this limit, the basic excitations of the worldsheet theory are solitons known as "Giant Magnons" which propagate on an infinitely long string [5]. The exact ABAE lead to non-trivial predictions for the dispersion relation of these solitons and also for their scattering matrix. These predictions were compared to the results of a leading-order semiclassical calculation in [5] (see also [8, 9]).

The main aim of this paper is to extend this comparison by performing a first-principles calculation of the soliton dispersion relation and S-matrix [1] to the next order[1] in the semiclassical expansion of the worldsheet theory. Our main result is a complete calculation of the soliton S-matrix at one-loop, which yields exact agreement with the predictions of the ABAE. In particular, we will reproduce in full the Hernandez-Lopez (HL) term in the magnon S-matrix which was originally obtained by considering the one-loop quantum correction to a circular string in $AdS_5 \times S^5$ [6, 7]. Our calculation, therefore provides further confirmation of the *universality* of the HL term in semiclassical string physics on $AdS_5 \times S^5$. For other interesting recent work on one-loop corrections, including a derivation of the HL term in the context of finite gap solutions see [10, 11][2] (see also [12] and [13]). In the rest of this introductory section we will review some basic features of semiclassical soliton quantisation [14−16] required for our analysis.

For simplicity we begin by considering the theory of a single scalar field $\varphi(x,t)$ of mass m in one space and one time dimension with a dimensionless coupling constant $1/g$. The field obeys non-linear equations with a two-parameter family of soliton solutions,

$$\varphi = \varphi_{cl}(x,t;x^{(0)},p) \tag{1.1}$$

The soliton is a localised lump of energy density $\mathcal{E}(x,t)$ centred around the point $x = x^{(0)}$ at time $t = 0$ (see figure 1). The parameter p corresponds to the conserved momentum conjugate to the spatial coordinate x. The soliton has finite classical energy $E(p) = gE_{cl}(p)$ and moves at constant velocity $v = v(p) \sim dE/dp$. At time t, the energy density is therefore centred around the point $x = x^{(0)} + vt$. All these features are realised, for example, in the specific case of the sine-Gordon kink. To match as closely as possible the case of interest, we will *not* assume $(1+1)$-dimensional Lorentz invariance for the full non-linear equations of motion.[3] Thus the solution $\varphi_{cl}(x,t;x^{(0)},p)$ is not related in a simple way to the solution with $p = 0$. However, again motivated by the specific problem of interest, we will assume that the the *linearised* equation of motion takes the standard relativistic form $(-\partial_t^2 + \partial_x^2 + m^2)\varphi(x,t) = 0$. It follows that the soliton configuration has exponentially decaying asymptotics at left and right spatial infinity,

$$\varphi_{cl}(x,t;x^{(0)},p) \sim \exp(-c|x|) \qquad \text{as} \qquad x \to \pm\infty\,, \tag{1.2}$$

where $c = c(p)$ is a positive constant which is equal to the mass m for a static soliton at rest.

After quantisation, the soliton yields a massive single-particle asymptotic state of the theory. Its dispersion relation has a semiclassical expansion of the form,

$$E(p) = gE_{cl}(p) \; + \; \Delta E(p) \; + \; O\left(\frac{1}{g}\right). \tag{1.3}$$

[1]In the following, we will refer to the first two orders in the semiclassical expansion as tree-level and one-loop respectively.

[2]We comment further on the relation of our calculation to the approach of these references at the end of this section.

[3]In the case of the string world-sheet theory in static gauge, two dimensional Lorentz invariance is broken by the Virasoro constraints.

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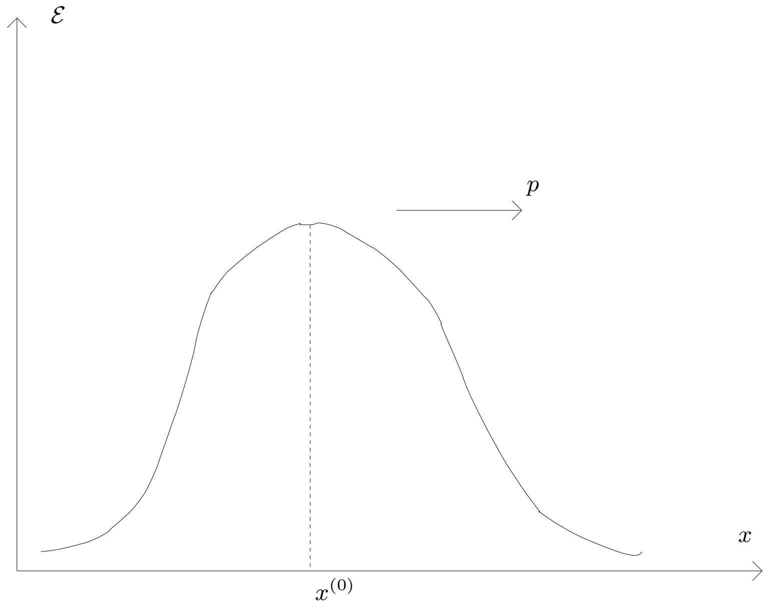

Figure 1: The one-soliton solution at time $t = 0$.

Our first goal is to calculate the one-loop correction to the energy $\Delta E(p)$. In general, one-loop quantum corrections are determined by the spectrum of the small fluctuation operator,

$$\hat{H} = \left. \frac{\delta^2 \mathcal{L}}{\delta\varphi(x,t)^2} \right|_{\varphi=\varphi_{cl}(x,t;x^{(0)},p)} \tag{1.4}$$

where $\mathcal{L}$ is the Lagrangian density of the theory. In particular we will study the auxiliary Schrodinger problem defined by the linearised equation of motion in the soliton background,

$$\hat{H}\psi(x,t) = 0 \tag{1.5}$$

where we will consider complex solutions $\psi \in \mathbb{C}$.

The asymptotics of $\hat{H}$ are determined by the asymptotics of the soliton solution to be,

$$\hat{H} \rightarrow -\partial_t^2 + \partial_x^2 + m^2 \quad + \quad O(e^{-c|x|}) \tag{1.6}$$

for $x \rightarrow \pm\infty$ at fixed time t. For each $k \in \mathbb{R}$, we can choose a solution, $\psi_k(x,t)$ of the small fluctuation equation (1.5) which goes like,

$$\psi(x,t) \sim \exp\left(iE(k)t + ikx\right) \tag{1.7}$$

with $E(k) = \sqrt{k^2 + m^2}$, as $x \rightarrow -\infty$. This corresponds to a plane-wave with wave number k incident upon the soliton from the left. Following the same solution to the asymptotic region to the right of the soliton, we will find that the solution will consist of a transmitted wave of the form,

$$\psi(x,t) \sim \exp\left(i\delta(k;p)\right) \exp\left(iE(k)t + ikx\right) \tag{1.8}$$

as $x \to +\infty$, where the real quantity $\delta(k;p)$ corresponds to the phase shift due to scattering on the soliton background. Of course in a general scattering problem, to obtain asymptotics of the form (1.8) we would also have to include a reflected wave which modifies the left asymptotics (1.7). A special feature of many integrable partial differential equations with soliton solutions and, in particular, of the cases considered in this paper, is that the classical reflection amplitude vanishes. Another potential complication is the existence of normalisable bound state solutions of the linearised equation (1.5) with exponentially decaying asymptotics. Again, this feature will be absent in all the cases considered in this paper.

The quantity $\delta(k;p)$ describes the *classical* scattering of a plane wave off the soliton background. As we now review, this classical scattering data is the basic ingredient we need to compute one-loop *quantum* corrections to the soliton. In particular, the phase shift $\delta(k;p)$ determines the density of scattering states which provides the measure for integrating over the continuous spectrum of the small fluctuation operator $\hat{H}$. The resulting formula for the one-loop correction to the soliton energy is [14],

$$\Delta E(p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\partial\delta(k;p)}{\partial k}\sqrt{k^2+m^2}\,. \tag{1.9}$$

The derivation of this formula is given in appendix A.

In the following we will need a slight generalisation to the case of N_F scalar fields φ_I, $I = 1,2,\ldots,N_F$, with Bose/Fermi statistics depending on the sign $(-1)^{F_I}$. We will assume that fluctuations of each these fields around the soliton background have the same asymptotic dispersion relation $E = \sqrt{k^2+m^2}$ and that the classical scattering matrix of the fluctuations is diagonal with eigenvalues $\exp(i\delta_I(k;p))$, $I = 1,2,\ldots,N_F$. All these features will be present in the case of interest below. With these assumptions, the one-loop correction to the dispersion relation becomes,

$$\Delta E(p) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{+\infty} dk \frac{\partial\delta_I(k;p)}{\partial k}\sqrt{k^2+m^2}\,. \tag{1.10}$$

In general the formulae (1.9), (1.10) may suffer from UV divergences which require regularisation. In the supersymmetric case of interest, we will find that these divergences cancel between Bosons and Fermions.

A characteristic feature of integrable PDEs in two spacetime dimensions is the existence of exact classical solutions describing the scattering of an arbitrary number of solitons. Here we will focus on a solution describing the scattering of two solitons of momenta p_1 and p_2 (see figure 2),

$$\varphi = \varphi_{\rm scat}(x,t;x_1^{(0)},x_2^{(0)},p_1,p_2)\,. \tag{1.11}$$

As shown, the solution also depends on the positions $x_1^{(0)}$ and $x_2^{(0)}$ of the two constituent solitons at time $t=0$. The conservation of the higher conserved charges implied by integrability ensures that the only effect of the scattering is a time delay $\Delta T(p_1,p_2)$ relative to free propagation of the two constituent solitons. Thus, in the far past $t \to -\infty$, and the

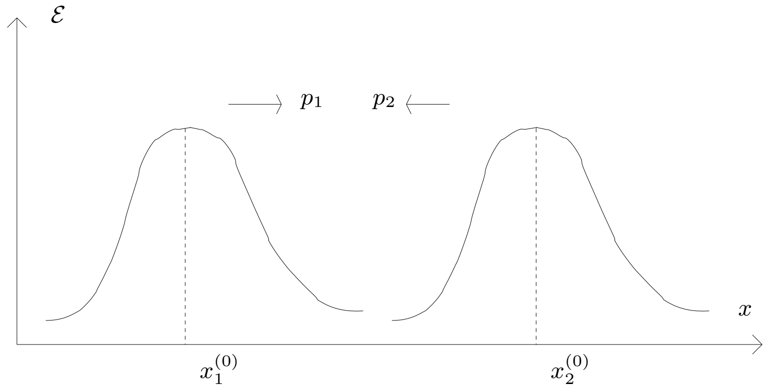

Figure 2: The two-soliton scattering solution at time $t = 0$.

far future, $t \to +\infty$ the solution asymptotes to a linear superposition of two single soliton solutions,

$$\varphi_{\rm scat}(x, t; x_1^{(0)}, x_2^{(0)}, p_1, p_2) \to \varphi_{cl}(x, t; x_1^{\pm}, p_1) + \varphi_{cl}(x, t; x_2^{\pm}, p_2) \tag{1.12}$$

where the asymptotic values of the position parameters[4] as $t \to \pm\infty$ are,

$$x_1^{\pm} = x_1^{(0)} \mp v_1 \frac{\Delta T}{2}, \qquad x_2^{\pm} = x_2^{(0)} \mp v_2 \frac{\Delta T}{2} \tag{1.13}$$

and, as above, the individual soliton velocities are $v_i \sim dE_i/dp_i$ for $i = 1, 2$.

Another important consequence of integrability is the factorisation of the scattering data corresponding to the two soliton solution. In particular, a plane-wave of wave number k incident on the two-soliton solution from the left experiences a phase shift,

$$\delta(k; p_1, p_2) = \delta(k; p_1) + \delta(k; p_2). \tag{1.14}$$

In other words, the phase shift experienced by the incident wave is simply the sum of the phase-shifts associated with the two constituent solitons. This property, which we will verify directly below, is related to the complete factorisation of the S-matrix which is a hallmark of an integrable theory.

In the quantum theory, solitons correspond to asymptotic states which scatter with an S-matrix,

$$\mathcal{S}(p_1, p_2) = \exp\left(i\Theta(p_1, p_2)\right). \tag{1.15}$$

At weak coupling, $1/g \ll 1$, the phase Θ has a semiclassical expansion of the form,

$$\Theta(p_1, p_2) = g\Theta_{cl}(p_1, p_2) \; + \; \Delta\Theta(p_1, p_2) \; + \; O\left(\frac{1}{g}\right). \tag{1.16}$$

A famous formula of Jackiw and Woo [17] relates the leading semiclassical contribution to the quantum S-matrix and the time-delay $\Delta T(p_1, p_2)$ in classical scattering,

$$\Theta_{cl}(p_1, p_2) = \frac{1}{g} \int_{E_{\rm Th}}^{E(p_1, p_2)} \Delta T(E) dE \tag{1.17}$$

[4]The parameters $x_1^{\pm}$ and $x_2^{\pm}$ as defined in this equation should not be confused with the spectral parameters introduced later in the paper.

where $E_{\rm Th}$ denotes the threshold energy for scattering. Much less well known, is the simple formula which determines the one-loop correction to the S-matrix in an integrable field theory in terms of the classical scattering data,

$$\Delta\Theta(p_1,p_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\partial\delta(k;p_1)}{\partial k}\,\delta(k;p_2)\,. \tag{1.18}$$

This formula was first obtained in the context of sine-Gordon theory by Faddeev and Korepin [15].[5] A more general derivation is provided in appendix B. Again we will also require a generalisation to the case of N_F fields with diagonal scattering (see equation (1.10)),

$$\Delta\Theta(p_1,p_2) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{+\infty} dk \frac{\partial\delta_I(k;p_1)}{\partial k}\,\delta_I(k;p_2)\,. \tag{1.19}$$

The above formulae (1.10) and (1.19) reduces the problem of computing one-loop corrections to the soliton dispersion and S-matrix to the problem of finding the *classical* phase shifts, $\delta_I(k;p)$, of small fluctuations around the background of a *single* soliton. The bulk of the paper is devoted to solving this problem for the case of a Giant Magnon solution of the worldsheet theory of arbitrary charge. In fact, we will describe three different approaches to determining the phase shifts. The first method is originally due to a clever observation of Dashen, Hasslacher and Neveu [14], that a linearised fluctuation around a background containing n solitons can be obtained as a degenerate limit of an $n+1$ soliton solution. In particular we will apply this approach to the exact multi-soliton solutions of the bosonic world sheet fields constructed via the dressing method developed in [18, 19]. The second approach relies on obtaining the spectral data for fluctuations around the Giant Magnon in the finite-gap formalism of [20, 21]. This approach reproduces the results of the dressing method for the bosonic fluctuations and also produces explicit results for the phase shifts of the fermionic fields. Finally, we provide a further check on the phase shifts by comparing them with the proposed exact magnon S-matrix [3] in a limit where one magnon becomes a worldsheet soliton and the other becomes an elementary fluctuation of the worldsheet fields. Note that this comparison involves only the leading-order piece of the proposed exact S-matrix in the semiclassical limit which already has many independent tests. Having extracted the classical scattering data, we use it to calculate the one-loop correction to the S-matrix of two giant magnons using formula (1.19) and compare with the Hernandez-Lopez one-loop contribution [6] to the exact S-matrix. We also demonstrate the vanishing of the one-loop correction to the soliton energy, completing an earlier partial calculation appearing in [42].

The paper is organised as follows. In the next section we review the predictions for soliton scattering coming from the ABAE. In section 3 we describe the different approaches to extracting the classical scattering data outlined above. Finally in section 4 we complete the calculation of the one-loop corrections to the soliton dispersion relation and S-matrix obtaining exact agreement with the predictions described in section 2. Various technical details and derivations are relegated to the appendices.

[5] See, in particular, Eqn (4.6) on p62 of this reference and the discussion following Eqn (4.28) on p66.

In interesting recent work, Gromov and Vieira [10, 11] have also provided a semiclassical derivation of the Hernandez-Lopez phase. Our calculation differs from theirs in that we are directly computing the S-matrix for soliton scattering with vacuum boundary conditions, while they are computing the one-loop energy shift for finite gap solutions with periodic boundary conditions. Nevertheless it is clear that the two calculations are related. In particular, in section 3.2, we obtain the classical scattering data for the fermionic worldsheet fields in the soliton background by taking a limit of an appropriate finite gap solution. On the other hand, the scattering data for the bosonic worldsheet fields is obtained in section 3.1 by explicit construction of soliton scattering solutions.

2. Predictions from Bethe Ansatz and Scattering Matrix

2.1 The asymptotic spectrum and its semiclassical limits

The asymptotic spectrum of the gauge theory spin chain consists of an infinite tower of BPS states labelled by a positive integer Q, $Q \in \mathbb{Z}^+$, and their conserved momentum p. The elementary excitation, called the "magnon", corresponds to the case $Q = 1$. States with $Q > 1$ correspond to the bound states of these elementary magnons [22]. Being short representations of the extended residual symmetry algebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ which carry conserved central charges, the dispersion relation of the elementary magnon and the bound states is then fixed by the shortening condition to be [3, 4, 22, 29],

$$\Delta - J = E = \sqrt{Q^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)}. \tag{2.1}$$

Here we have introduced the coupling $g^2 = \lambda/16\pi^2$. The magnon dispersion relation (2.1) is common to all states in the supermultiplet of dimension $16Q^2$ [23].

As usual we introduce a convenient representation of the dispersion relation in terms of spectral parameters $X^\pm$ where,

$$p(X^\pm) = \frac{1}{i}\log\left(\frac{X^+}{X^-}\right) \; ; \tag{2.2}$$

so that the energy E and charge Q can be expressed as

$$E(X^\pm) = \frac{g}{i}\left[\left(X^+ - \frac{1}{X^+}\right) - \left(X^- - \frac{1}{X^-}\right)\right], \tag{2.3}$$

$$Q(X^\pm) = \frac{g}{i}\left[\left(X^+ + \frac{1}{X^+}\right) - \left(X^- + \frac{1}{X^-}\right)\right]. \tag{2.4}$$

Real values of E and P are obtained by imposing $X^- = (X^+)^*$. In the following we will use lower case letters $x^\pm$ and $y^\pm$ to denote the spectral parameters in the case of the elementary magnon $Q = 1$.

It will be of interest to understand the semiclassical string limit, $g \to \infty$ of the elementary magnons and their bound states. Importantly, there are several distinct ways in

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which the limit can be taken. The first, which we will call the "plane-wave" limit[6] is given by:

$$g \to \infty\,, \quad p \sim \frac{1}{g}\,, \quad Q \;\; \text{Fixed}\,. \tag{2.5}$$

In terms of the spectral parameters $X^{\pm}$, this can be equivalently imposed by

$$X^+ \sim X^- \approx r + \mathcal{O}(1/g)\,, \qquad r \in \mathbb{R}\,, \tag{2.6}$$

so that the dispersion relation for the magnon and its bound states becomes

$$\Delta - J = \sqrt{Q^2 + k^2}\,, \tag{2.7}$$

where the combination $k = 2gp \sim \mathcal{O}(g^0)$ is kept fixed in the limit (2.6). In this limit, the magnon goes over to the elementary excitation of the worldsheet fields of the $AdS_5 \times S^5$ string. In canonical quantisation these states are the quanta associated with linearised fluctuations of the worldsheet fields around a point-like string (the BMN solution [25]) which orbits the equator of S^5 at the speed of light. The fluctuations take the form of plane-waves which solve the linearised equations of motion of the worldsheet theory and have the form,

$$\delta Z(x,t) \sim \exp(i\omega t + ikx)\,, \tag{2.8}$$

where Z is a complex worldsheet field. The frequency $\omega = \sqrt{1+k^2}$ and the wave number k can also be written in term of the magnon spectral parameters $x^{\pm} \simeq r$ as

$$\omega(r) = \frac{r^2+1}{r^2-1}\,, \quad k(r) = \frac{2r}{r^2-1}\,. \tag{2.9}$$

States with $Q > 1$ correspond to bound states of the elementary worldsheet excitations in this limit.

A second interesting limit is the so-called "Giant Magnon" limit [5] which corresponds to

$$g \to \infty\,, \quad p \;\; \text{Fixed}\,; \tag{2.10}$$

for a BPS state of fixed charge Q. Equivalently, in terms of the corresponding spectral parameters we have,

$$X^+ \sim \frac{1}{X^-} \approx \exp(ip/2) + \mathcal{O}(1/g)\,. \tag{2.11}$$

In this limit the spin-chain magnon and its bound states correspond to a classical soliton configuration on the string worldsheet with dispersion,

$$\Delta - J \approx 4g \left|\sin\left(\frac{p}{2}\right)\right| \;\; + \;\; O(1/g)\,. \tag{2.12}$$

The corresponding string energy $E = \Delta - J$ scales linearly in g as appropriate for a classical soliton. In the target spacetime, the worldsheet soliton is identified with loop of open string

[6]This limit takes its name from its relation to the Penrose limit where the dual string background becomes a gravitational plane-wave. In the following we will see that the terminology is also appropriate for an unrelated reason, namely that the magnon is naturally associated with the plane-wave solutions of the linearized equation of motion in this limit.,BMN

with endpoints on an equator of S^5. It is interesting to note that there is no $\mathcal{O}(g^0)$ term in the expansion of the exact dispersion relation (2.1) in this limit. This indicates that the soliton energy does not receive a correction at one-loop order in the semiclassical expansion, which corresponds to an expansion in powers of $1/g$.

Although the magnon looks quite different in the plane-wave and giant magnon limits described above, it is possible to smoothly interpolate between the two cases. The elementary quantum of the worldsheet theory and the classical soliton are representatives of the same excitation in different regions of momentum space. This is particularly clear if one considers the "near flat-space" limit introduced [26] where $x^+ \sim x^- \sim 1$.

The Giant Magnon limit discussed above is identical for all BPS states of fixed charge Q. As the charge is an adjustable parameter we can also take a different limit,

$$g \to \infty\,, \quad Q \sim g\,, \quad p \;\; \text{Fixed}\,; \tag{2.13}$$

where the spectral parameters $X^\pm$ remain fixed and, as before, obey the constraint:

$$\left(X^+ + \frac{1}{X^+}\right) - \left(X^- + \frac{1}{X^-}\right) = i\frac{Q}{g} \sim \mathcal{O}(g^0)\,. \tag{2.14}$$

This limit yields a family of classical soliton configurations of the worldsheet theory, with energy

$$\Delta - J = \sqrt{Q^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \sim \mathcal{O}(g)\,, \tag{2.15}$$

where $Q \sim g$ is now regarded as a continuous parameter.[7] These solutions are known as "Dyonic Giant Magnons" (DGMs) [27] (see also [18, 28, 44]) and we will refer to the corresponding limit as the "DGM limit". The previously discussed Giant Magnon of [5] is obtained as a smooth $Q \to 0$ of this more general solution. As the DGM dispersion relation coincides with the exact dispersion relation (2.1), the only correction is the quantisation of the charge Q integer units. As in the ordinary Giant Magnon case, we should therefore expect that the one-loop correction to the soliton energy vanishes. In the following we will check this vanishing by a direct calculation in the worldsheet theory.

2.2 The magnon scattering matrix

The exact S-matrix for two elementary magnons in the same $\mathfrak{su}(2)$ sub-sector takes the form,

$$s_{\mathfrak{su}(2)}(x^\pm, y^\pm) = s_{\text{BDS}}(x^\pm, y^\pm)\sigma^2(x^\pm, y^\pm)\,, \qquad s_{\text{BDS}}(x^\pm, y^\pm) = \frac{x^+ - y^-}{x^- - y^+}\frac{1 - 1/x^+y^-}{1 - 1/x^-y^+}\,. \tag{2.16}$$

Here the factor $s_{\text{BDS}}(x^\pm, y^\pm)$ originates in the conjectured all-loop Bethe Ansatz of [29], and $\sigma(x^\pm, y^\pm) = \exp(i\theta(x^\pm, y^\pm))$ is known as the "dressing factor"and $\theta(x^\pm, y^\pm)$ will be called the "dressing phase". An exact form for the dressing phase was recently conjectured in [30]. Following earlier important work [31], the authors implemented the constraints

[7] Like all the soliton solutions considered here the solution also depends non-trivially on the momentum p and the initial position $x^{(0)}$.

of crossing symmetry [32] and Kotikov-Lipatov's principle of maximal transcendentality to obtain an explicit expression for the phase [33, 34] (see also [35] for earlier proposal using transcendentality principle). The poles of the resulting magnon S-matrix correspond precisely with expectations based on the exact spectrum (2.1) [24]. In the strong coupling, expansion, the conjectured phase of [30] reproduces the previously obtained tree-level [36] and the one-loop [6] contributions.

The exact dressing phase $\theta(x^\pm, y^\pm)$ is anti-symmetric under the interchanges of spectral parameters and can be written as,

$$\theta(x^\pm, y^\pm) = k(x^+, y^+) - k(x^+, y^-) - k(x^-, y^+) + k(x^-, y^-)\,. \tag{2.17}$$

In the strong-coupling limit, $g \to \infty$, $\theta(x^\pm, y^\pm)$ and $k(x, y)$ can be expanded as,

$$\theta(x^\pm, y^\pm) = g\theta_0(x^\pm, y^\pm) + \theta_1(x^\pm, y^\pm) + \mathcal{O}(1/g)\,, \tag{2.18}$$

$$k(x, y) = gk_0(x, y) + k_1(x, y) + \mathcal{O}(1/g)\,. \tag{2.19}$$

respectively. The explicit form of the tree-level contribution was first proposed in [36] and take the form,

$$k_0(x, y) = \left[\left(y + \frac{1}{y}\right) - \left(x + \frac{1}{x}\right)\right] \log\left(1 - \frac{1}{xy}\right)\,. \tag{2.20}$$

The one-loop term $k_1(x, y)$ was first obtained in [6] from considering the quantum fluctuations certain spinning string solution and can be written as [37],

$$k_1(x, y) = \kappa_1(x, y) - \kappa_1(y, x)\,, \tag{2.21}$$

$$\begin{aligned}\kappa_1(x, y) = &\frac{1}{\pi} \log\left(\frac{y-1}{y+1}\right) \log\left(\frac{x - 1/y}{x - y}\right)\\ &+\frac{1}{\pi}\Bigg[\mathrm{Li}_2\left(\frac{\sqrt{y} - 1/\sqrt{y}}{\sqrt{y} - \sqrt{x}}\right) - \mathrm{Li}_2\left(\frac{\sqrt{y} + 1/\sqrt{y}}{\sqrt{y} - \sqrt{x}}\right)\\ &\qquad + \mathrm{Li}_2\left(\frac{\sqrt{y} - 1/\sqrt{y}}{\sqrt{y} + \sqrt{x}}\right) - \mathrm{Li}_2\left(\frac{\sqrt{y} + 1/\sqrt{y}}{\sqrt{y} + \sqrt{x}}\right)\Bigg]\,.\end{aligned} \tag{2.22}$$

In the following our main concern will be with the consequences of the above expressions for the semiclassical scattering of worldsheet solitons. In particular, in the limit $g \to \infty$, the expression (2.20) determines the leading semiclassical contribution to the S-matrix of two Giant Magnons. This prediction was checked against a first-principles calculation in [5]. One of the main aims of this paper is to extend this check to the next order in $1/g$. In this regard, it is important to note that the correspondence between the expansions (2.18) and (2.19) and the semiclassical expansion of the worldsheet theory is not quite straightforward. The reason is that the magnon spectral parameters contain hidden dependence on the coupling g because of the constraint,

$$\left(x^+ + \frac{1}{x^+}\right) - \left(x^- + \frac{1}{x^-}\right) = \frac{i}{g} \tag{2.23}$$

which follows from (2.4) with $Q = 1$. This problem is easily avoided by working in the slightly more general context of the scattering of two magnon bound states of charges Q_1

and Q_2. As discussed in [8], the exact bound state S-matrix can be constructed from the exact magnon S-matrix via the standard fusion procedure. The result is conveniently expressed in terms of the bound state spectral parameters introduced above as,

$$S_{\mathfrak{su}(2)}(X^{\pm}, Y^{\pm}) = S_{\rm BDS}(X^{\pm}, Y^{\pm})\sigma^2(X^{\pm}, Y^{\pm})\,. \tag{2.24}$$

Here $S_{\rm BDS}(X, Y)$ is the exact expression constructed from applying the fusion procedure to the BDS part, $s_{\rm BDS}(x, y)$, of the magnon S-matrix in (2.16). The explicit expression, which will not be needed here, can be found in [8]. Importantly, the factor $\sigma^2(X^{\pm}, Y^{\pm})$ is exactly the *same* dressing factor appearing in the elementary magnon S-matrix (2.24), the only difference being that the bound state spectral parameters $X^{\pm}, Y^{\pm}$ replace the spectral parameters $x^{\pm}, y^{\pm}$ of the fundamental magnons.

We can now take the DGM limit (2.13) for both magnon bound states. As the spectral parameters remain fixed in this limit, the terms in the strong-coupling expansion of the dressing phase $\sigma(X^{\pm}, Y^{\pm})$ correspond directly to terms in the semiclassical expansion of the worldsheet theory. The resulting semiclassical S-matrix can be written in the first two orders as,

$$S_{\mathfrak{su}(2)}(X^{\pm}, Y^{\pm}) \cong \exp(2i\Theta(X^{\pm}, Y^{\pm}))\,, \tag{2.25}$$

$$\Theta(X^{\pm}, Y^{\pm}) = K(X^+, Y^+) - K(X^+, Y^-) - K(X^-, Y^+) + K(X^-, Y^-)\,, \tag{2.26}$$

$$K(X, Y) = gK_0(X, Y) + K_1(X, Y) + \mathcal{O}(1/g)\,. \tag{2.27}$$

The function $K_0(X, Y)$ was calculated in [8] and checked against a leading order semiclassical calculation of the dyonic giant magnon scattering matrix. It is given by

$$K_0(X, Y) = \left[\left(X + \frac{1}{X}\right) - \left(Y + \frac{1}{Y}\right)\right]\log(X - Y)\,. \tag{2.28}$$

Notice that $K_0(X, Y)$ is functionally different from $k_0(x, y)$ in (2.20). As explained in [8], this is due to a non-trivial contribution from the BDS piece $S_{\rm BDS}$. At the next order, we have

$$K_1(X, Y) = k_1(X, Y)\,, \tag{2.29}$$

where the function k_1 is defined in (2.21). In other words the one-loop contribution to the bound state S-matrix comes purely from the dressing phase and is therefore functionally identical to the Hernandez-Lopez contribution to the magnon dressing phase. This can be traced to the fact that the BDS term $S_{\rm BDS}(X^{\pm}, Y^{\pm})$ is analytic in g^2 and therefore only contributes at even loop order in the worldsheet expansion.

The main conclusion of this section concerns the predictions for the one-loop contributions to the dispersion relation and scattering matrix of Dyonic Giant Magnons. Specifically we have seen that the known exact dispersion relation requires that the one-loop correction to the soliton mass vanishes exactly. The one-loop correction to the S-matrix can be expressed in terms of the Dyonic Giant Magnon spectral parameters $X^{\pm}$ and $Y^{\pm}$ defined above and is functionally identical to the Hernandez-Lopez contribution to the magnon dressing phase. In the rest of the paper we will test these results against direct semiclassical calculations.

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3. Determining the classical phase shifts

As explained in the introduction, the main ingredient in the calculation of one-loop quantum corrections is the classical scattering data for small fluctuations around the soliton solution. In particular we need to determine the phase shifts for classical plane waves scattering off multiple solitons. In this section we will address this problem using three different approaches, each of which will yield part of the information we need.

The starting point is the Metsaev-Tseytlin action for the Green-Schwarz superstring in $AdS_5 \times S^5$ in conformal gauge [38 − 40]. Here the global embedding that parametrises the $AdS_5 \times S^5$ spacetime can be chosen as

$$\begin{aligned} AdS_5: &\quad -|Y_1|^2 + |Y_2|^2 + |Y_3|^2 = -1\,, \\ S^5: &\quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1\,. \end{aligned} \tag{3.1}$$

For our study of worldsheet scattering matrix, Y_1 and Z_1 are gauge-fixed to form the longitudinal light-cone coordinates, whereas $\{Y_2, \bar{Y}_2, Y_3, \bar{Y}_3; Z_2, \bar{Z}_2, Z_3, \bar{Z}_3\}$ become eight bosonic transverse excitations and combine to transform in the (bi-)vector representation under the residual $SO(4) \times SO(4)$ subgroup. Similarly for the worldsheet fermions, under such gauge choice, the remaining components (after fixing κ-symmetry) become $\{\theta_1, \ldots, \theta_4, \eta_1, \ldots, \eta_4\}$, they combine to transform in the bi-spinor representation of $SO(4) \times SO(4)$. Together, the eight bosonic and eight fermionic fluctuations form the bi-fundamental representations of residual $P\mathrm{SU}(2|2)^2$ symmetry group. We will consider these sixteen fluctuations around a classical soliton background, and we shall use a uniform notation to denote them:

$$\mathcal{I} \equiv \{\overbrace{Y_2, \bar{Y}_2, Y_3, \bar{Y}_3}^{\mathcal{I}_{AdS_5}}; \overbrace{Z_2, \bar{Z}_2, Z_3, \bar{Z}_3}^{\mathcal{I}_{S^5}}; \underbrace{\theta_1, \theta_2, \theta_3, \theta_4; \eta_1, \eta_2, \eta_3, \eta_4}_{\mathcal{I}_{\text{fermions}}}\}.$$

As we review below, the Dyonic Giant Magnon (DGMs) is a soliton solution of the worldsheet theory for which the corresponding string motion occurs on an $\mathbb{R} \times S^3$ submanifold of $AdS_5 \times S^5$. We need to consider linearised fluctuations of all of the world sheet fields around the classical solution corresponding to one or more DGMs. The necessary phase shifts are then encoded in the asymptotics of the fluctuations in the limits $x \to \pm\infty$ where x is the space-like worldsheet coordinate. In the next subsection we will proceed by constructing multi-DGM solutions and their classical fluctuation spectra explicitly using the dressing method. In its present form this method is only applicable to the bosonic worldsheet fields. In subsection 3.2, we employ a different method based on the finite gap construction of [20] which also yields the phase shifts for the fermionic worldsheet fields. Finally, in subsection 3.3, we describe a third method using the proposed all-loop magnon scattering matrix [3] which provides further non-trivial checks on our results.

3.1 Phase shifts from the dressing method

In this section we present the semiclassical phase shifts calculated directly from string sigma model using the so-called *"dressing method"*. This is a standard technique for constructing multi-soliton solutions in classical integrable systems which was applied in the present context by Spradlin and Volovich [18].

As discussed in the previous section, the Dyonic Giant Magnon (DGM) is a soliton on the string worldsheet [27]. It corresponds to a family of classical solutions labelled by the conserved momentum p and charge $Q = J_2$, where J_2 is one of the three generators J_1, J_2, J_3 for the global symmetry group $SO(6)$ of the sphere S^5. This data can be equivalently given by two complex spectral parameters X^+ and $X^- = (X^+)^*$. The solution is also labelled by its initial position $x^{(0)}$ as well as some extra parameters which determine its orientation inside S^5 at time $t = 0$. As discussed above, the DGM admits a special limit where $X^+ \simeq 1/X^-$, the charge vanishes and the solution reduces to an ordinary Giant Magnon of the type considered by Hofman and Maldacena [5]. It also admits a limit where $X^+ \simeq X^- \simeq r$ and it collapses to the vacuum. In the target space the vacuum configuration is just the BMN string solution describing a pointlike string orbiting an equator of S^5. Near this degenerate point the soliton solution reduces to a solution of the linearised equations of motion corresponding to a plane wave of small amplitude with wave number and frequency,

$$\omega(r) = \frac{r^2+1}{r^2-1}\,, \quad k(r) = \frac{2r}{r^2-1}\,. \tag{3.2}$$

As we shall review below, the dressing method allows us to construct exact multi-soliton solutions of the worldsheet theory. In particular we can construct a configuration containing N DGMs with individual spectral parameters $X_i^\pm$, for $i = 1, 2, \ldots, N$. We can now take a limit where, for example for the n-th DGM $X_n^+ \simeq X_n^-$ and the solution collapses to the one describing $N-1$ DGMs. Near this limit the exact solution must go over to a solution of the equations linearised around the $N-1$ soliton solution. As first noted by Dashen, Hasslacher and Neveu [14], this construction provides an elegant way of extracting the exact spectrum of small fluctuations and, in particular, the corresponding phase shifts. We will now apply this methodology to the bosonic sector of the worldsheet σ model. Some of the the calculation details are relegated to appendix C.

The Dyonic Giant Magnon corresponds to string motion in an $\mathbb{R} \times S^3$ subspace of the the full $AdS_5 \times S^5$ spacetime. It is easy to check that fluctuations in the AdS_5 directions couple trivially to this background and thus have vanishing phase shifts. Thus we will focus on the S^5 sector of the worldsheet theory. Following [18] we work in static gauge and the worldsheet theory in this sector essentially reduces to a bosonic sigma-model on a flat two-dimensional worldsheet with the coset $\mathrm{SU}(4)/\mathrm{Sp}(2) \approx S^5$ as target space. The equations of motion of this σ-model must be also supplemented by the Virasoro constraints.

The coset construction exploits the existence of a $\mathbb{Z}_2$-automorphism $\Omega \in \mathrm{Aut}(\mathrm{SU}(4))$, where

$$\Omega(\mathbf{g}) := J^{-1}\mathbf{g}^T J, \quad \text{where} \quad \mathbf{g} \in \mathrm{SU}(4) \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

It has the property $\Omega^2 = 1$. The condition

$$\Omega(\mathcal{P}) = \mathcal{P} \quad \text{for} \quad \mathcal{P} \subset \mathrm{SU}(4) \tag{3.4}$$

will give us $\mathcal{P} \approx \mathrm{SU}(4)/\mathrm{Sp}(2) \approx S^5$. This allows for a decomposition of SU(4) into

$$\mathrm{SU}(4) \approx \mathrm{Sp}(2) \otimes \mathrm{SU}(4)/\mathrm{Sp}(2)\,.$$

A convenient parametrisation of the coset is given by,

$$\mathbf{g} = \begin{pmatrix} Z_1 & Z_2 & 0 & Z_3 \\ -\bar{Z}_2 & \bar{Z}_1 & -\bar{Z}_3 & 0 \\ 0 & \bar{Z}_3 & Z_1 & -\bar{Z}_2 \\ -\bar{Z}_3 & 0 & Z_2 & \bar{Z}_1 \end{pmatrix},$$

where the components Z_1, Z_2 and Z_3 satisfy $\sum_{i=1}^{3} |Z_i|^2 = 1$. By defining the flat current $j = -\mathbf{g}^{-1} d\mathbf{g} \in \mathfrak{su}(4)$, we can make the following decomposition,

$$j = H + P, \quad H \in \mathfrak{sp}(2), \quad P \in \mathfrak{su}(4)/\mathfrak{sp}(2).$$

The equations of motion for the sigma model can then be written succinctly as

$$d \star P = \star P \wedge H + H \wedge \star P\,,$$

where $\star$ denotes the Hodge-dual with respect to the worldsheet metric. These can be equivalently be expressed as the zero curvature condition of the following flat Lax connection

$$\hat{j}(X) = H + \frac{1+X^2}{1-X^2} P + \frac{2X}{1-X^2} \star P,$$

with $X \in \mathbb{C}$ being a spectral parameter, and notice that $\hat{j}(X=0) = j$. By picking the coordinates $z_\pm = \frac{1}{2}(x \pm t)$ as coordinates in the worldsheet, we find this connection has the form

$$\hat{j}(X) = H + \frac{\partial_- \mathbf{g}\,\mathbf{g}^{-1}}{1-X} + \frac{\partial_+ \mathbf{g}\,\mathbf{g}^{-1}}{1+X}.$$

The flatness condition for $\hat{j}$ is equivalent to the consistency conditions for the auxiliary linear problem,

$$\Big[\partial_- - \frac{\partial_- \mathbf{g}\,\mathbf{g}^{-1}}{1-X}\Big]\Psi(X) = 0\,,$$
$$\Big[\partial_+ - \frac{\partial_+ \mathbf{g}\,\mathbf{g}^{-1}}{1+X}\Big]\Psi(X) = 0\,.$$

Clearly $\Psi(X=0) = \mathbf{g}$ will be a solution to these equations. Only those solutions that further obey (3.4) and the Virasoro constrains will be solutions of the string equations of motion.

The trivial vacuum solution of the equations of motion corresponding to the BMN point like string solution is given by,

$$\Psi_0(X) = \mathrm{diag}(e^{iZ(X)}, e^{-iZ(X)}, e^{iZ(X)}, e^{-iZ(X)}),$$

where

$$Z(X) = \frac{z_-}{X-1} + \frac{z_+}{X+1}\,,$$

This solution has vanishing energy $\Delta - J_1 = 0$.

The dressing method proceeds by applying a spectral-parameter dependent gauge transformation to both the connection $\hat{j}(X)$ and the auxiliary wave function $\Psi(X)$. It is a solution generating technique that can be used to map trivial solutions of the equations of motion into new non-trivial solutions. Here we review the construction given in [18, 19], more details can be found in [18, 19, 41]. Explicitly, a new solution can be determined from the vacuum solution by acting on it with a gauge transformation $\chi_1(X)$,

$$\begin{cases} \Psi_1(X) = \chi_1(X)\Psi_0(X)\,, \\ \hat{j}_1(X) = \chi_1(X)\,\hat{j}_0(X)\,\chi_1^{-1}(X) + d\chi_1(X)\,\chi_1^{-1}(X)\,, \end{cases}$$

where $\hat{j}_0(X) \equiv \hat{j}(X)|_{\mathbf{g}=\mathbf{g}_0}$ and

$$\chi_1(X) = 1 + \frac{X_1 - \bar{X}_1}{X - X_1}\mathcal{P}_1[w_1] + \frac{1/\bar{X}_1 - 1/X_1}{X - 1/\bar{X}_1}\mathcal{Q}_1[w_1]\,.$$

The projection operators $\mathcal{P}_1, \mathcal{Q}_1$ are determined from Ψ_0 itself by requiring that the dressing transformation does not change the analytic structure of the Lax connection $\hat{j}(X)$ and that $\Psi_1(X)$ obeys (3.4). Here w_1 is a four-component vector specifying the orientation of the solution in the target space. In particular, by taking $\mathbf{g}_1 = \Psi_1(0)$ and making the identifications $X_1 = r_1 e^{ip_1/2} \equiv X_1^+$ and $\bar{X}_1 = r_1 e^{-ip_1/2} \equiv X_1^-$, and selecting the polarisation vector $w_1 \equiv w_\parallel = (1, i, 0, 0)^t$, we recover the familiar DGM solution of [27],

$$\mathbf{g}_1 = \mathbf{g}_0 - \frac{X_1^+ - X_1^-}{X_1^+}\Big(\mathcal{P}_1[w_1] + \mathcal{Q}_1[w_1]\Big)\mathbf{g}_0\,.$$

This solution has the following conserved quantities:

$$\Delta - J_1 = 2g\frac{1 + r_1^2}{r_1}\Big|\sin\Big(\frac{p_1}{2}\Big)\Big|\,, \tag{3.5}$$

$$J_2 = 2g\frac{1 - r_1^2}{r_1}\Big|\cos\Big(\frac{p_1}{2}\Big)\Big|\,, \tag{3.6}$$

$$J_3 = 0\,. \tag{3.7}$$

The orientation vector $w_1 = w_\parallel$ determines which $SU(2) \simeq S^3$ subspace of the target S^5 in which the DGM is embedded. Picking an orthogonal orientation vector $w_1 \equiv w_\perp = (i, 0, 0, 1)^t$ simply has the effect of interchanging the Cartan charges J_2 and J_3 and selecting a different $SU(2)$ subspace for the embedding.

As mentioned above, in the limit

$$p_1 \to 0, \quad r_1 \,\text{fixed} \Leftrightarrow X_1^+ \sim X_1^-$$

the DGM solution goes over to the vacuum, $\mathbf{g}_1(x,t) \to \mathbf{g}_0(x,t)$. Expanding $\mathbf{g}_1$ in $\eta \equiv X_1^+ - X_1^-$, we find that, at linear order in η, the resulting solution describes a plane wave propagating in the background described by $\mathbf{g}_0$. The dressing method allows us to determine easily an explicit expression for the perturbed solution by evaluating,

$$\mathbf{g}_1 = \mathbf{g}_0 + \delta\mathbf{g}_0,$$

with

$$\delta\mathbf{g}_0 = -2i\sin\Big(\frac{p_1}{2}\Big)\Big(\mathcal{P}_1[w_1] + \mathcal{Q}_1[w_1]\Big)\Big|_{\eta=0}\mathbf{g}_0,$$

being the plane-wave solution. For the orientation $w_1 \equiv w_\parallel$ we find,

$$\delta\mathbf{g}_0 \equiv \begin{pmatrix} \delta Z_1 & \delta Z_2 & 0 & \delta Z_3 \\ -\delta\bar{Z}_2 & \delta\bar{Z}_1 & -\delta\bar{Z}_3 & 0 \\ 0 & \delta\bar{Z}_3 & \delta Z_1 & -\delta\bar{Z}_2 \\ -\delta\bar{Z}_3 & 0 & \delta Z_2 & \delta\bar{Z}_1 \end{pmatrix} \tag{3.8}$$

$$= -2i\sin\Big(\frac{p_1}{2}\Big)\cdot\frac{1}{2}\begin{pmatrix} 1 & ie^{iv_1} & 0 & 0 \\ -ie^{iv_1} & 1 & 0 & 0 \\ 0 & 0 & 1 & ie^{iv'_1} \\ 0 & 0 & -ie^{iv'_1} & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & e^{-it} & 0 & 0 \\ 0 & 0 & e^{it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{pmatrix}, \tag{3.9}$$

where $v_1 \equiv Z(r_1) + \bar{Z}(r_1) = 2Z(r_1)$ and $v'_1 = v_1(1/r_1)$. We then obtain,

$$\delta Z_1 = -i\sin\Big(\frac{p_1}{2}\Big)\, e^{+it}\,. \tag{3.10}$$

$$\delta Z_2 = \sin\Big(\frac{p_1}{2}\Big)\, e^{i\omega_1 t - ik_1 x}\,, \tag{3.11}$$

$$\delta Z_3 = 0\,. \tag{3.12}$$

Thus the perturbation has the form of a plane wave with wave number given by $k_1 = 2r_1/1 - r_1^2$ and frequency $\omega_1 = 1 + r_1^2/1 - r_1^2 = \sqrt{k_1^2+1}$. As the background is the trivial vacuum there is no phase shift. We can also take an orthogonal orientation vector $w_1 = w_\perp$ to obtain identical results but with δZ_2 and δZ_3 interchanged.

We can now apply the same technique to determine the solution describing the propagation of a plane-wave in the n-soliton background. Since we merely need to determine the phase shifts $\delta_{Z_k}(r;\{X_j^\pm\})$ and $\delta_{\bar{Z}_k}(r;\{X_j^\pm\})$ corresponding to the fields δZ_k and $\delta\bar{Z}_k$, we will be only interested in the asymptotic limits of this perturbation solution rather than the full solution. The phase shifts in general can then be calculated from:

$$\delta_{Z_k}(r;\{X_j^\pm\}) = -i\log\left(\delta Z_k\right)\Big|_{+\infty} - i\log\left(\delta Z_k\right)\Big|_{-\infty}\,. \tag{3.13}$$

Here we only list the results calculated from this approach, and we present the relevant calculation details in the appendix C. The polarisations within this sector will be labelled by the coordinates that suffer a non-trivial phase shift, $I \in \mathcal{I}_{S^5} \equiv \{Z_2, \bar{Z}_2, Z_3, \bar{Z}_3\}$, i.e., a plane-wave aligned with the background soliton will have a non-trivial phase-shift in the directions $Z_2, \bar{Z}_2$, whether a plane-wave with a perpendicular polarisation will have a phase shift for $Z_3, \bar{Z}_3$.

$$\delta_{Z_2}\left(r;\{X_j^\pm\}\right) = -\delta_{\bar{Z}_2}\left(1/r;\{X_j^\pm\}\right) = -2i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) - P, \tag{3.14}$$

$$\delta_{Z_3}\left(r;\{X_j^\pm\}\right) = \delta_{\bar{Z}_3}\left(r;\{X_j^\pm\}\right) = -i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) - i\sum_{j=1}^{N}\log\left(\frac{1/r-X_j^-}{1/r-X_j^+}\right), \tag{3.15}$$

where $P \equiv \sum_{j=1}^{N} p_j$ is the total dyonic giant magnon momentum and $r = r(k)$ is related to the plane-wave momentum k by,

$$k = \frac{2r}{r^2 - 1}. \tag{3.16}$$

In the GM limit $X_j^{\pm} \to x_j^{\pm} \equiv \exp(\pm i p_j/2)$ the phase shifts take the form,

$$\delta_I(r; \{x_j^{\pm}\}) = -2i \sum_{j=1}^{N} \log\left(\frac{r - x_j^+}{r - x_j^-}\right) - P, \quad I \in \mathcal{I}_{S^5}. \tag{3.17}$$

Although the dressing method can not be directly applied to the fermionic case, a fermionic solution for a single Giant Magnon background was presented in [42] (See also earlier results in [43]). From there one easily determines the phase shift for the fermionic perturbations around an one-giant magnon soliton background with momentum $p = -i\log(x^+/x^-)$ as

$$\delta_I(r; x^{\pm}) = -i\log\left(\frac{r - x^+}{r - x^-}\right) - \frac{p}{2}, \quad I \in \mathcal{I}_{\text{fermions}} \equiv \{\theta_1, \ldots, \theta_4; \eta_1, \ldots, \eta_4\}. \tag{3.18}$$

As dictated by supersymmetry, the dispersion relation for a fermionic perturbation is identical to that of the bosons; $\omega = \sqrt{k^2+1}$ [18], with k the plane-wave momentum, related to r by (3.16).

3.2 Phase shifts from finite-gap solutions

In the previous section, by applying the dressing method to the S^5 sector, we were able to determine the phase shift caused by the scattering between a plane-wave bosonic fluctuation and a N-dyonic giant magnon soliton within certain $S^3 \subset S^5$. Extending the dressing method to the full theory including fermionic fluctuation remains an unsolved problem. In this subsection we will sidestep this difficulty by using another formalism [20] which allows us to construct the spectral data for solutions of the worldsheet σ-model with closed-string boundary conditions. In particular, the worldsheet fields are now taken to be periodic in the spatial coordinate x with period ℓ. In static gauge, where the energy density is constant along the string, the period is related to the string energy as $\ell = \Delta/2g$. We will consider string solutions with large but finite energy. Thus, for the moment, we are moving away from the strict Hofman-Maldacena limit described above where the string becomes infinitely long. For periodic boundary conditions the spectrum of fluctuations around a given classical background now becomes discrete. As we review below, the classical phase shift naturally appears in the corresponding quantisation condition for the wave number of the small fluctuations. If we pick a classical background which goes over to the DGM solution in the limit $\ell \to \infty$, we can then extract the required phase shifts for each worldsheet field.

3.2.1 Dyonic giant magnons as finite-gap solutions

We will begin this subsection by reviewing the elegant description of classical solutions with periodic boundary conditions obtained in [20] by Kazakov, Marshakov, Minahan and

Zarembo (KMMZ). To start with we will restrict our attention to states in a particular $SU(2)$ sector where the dual string motion is confined to an $\mathbb{R} \times S^3$ submanifold of the spacetime. As mentioned in the previous section equations of motion for the bosonic string admit a Lax formulation, with flat connection j, which immediately implies the existence of an infinite number of conserved charges at the classical level. The relevant classical solutions are naturally classified by the analytic behaviour of the corresponding monodromy matrix, $\Omega(X) = P\exp(\oint j)$, and its eigenvalues as functions of the complex spectral parameter $X \in \mathbb{C}$ introduced above. For classical strings on $\mathbb{R} \times S^3$, the monodromy matrix is a unimodular 2×2 matrix with eigenvalues $\exp(\pm ip(X))$. Here, the quasi-momentum $p(X)$ is a complex function of the spectral parameter with prescribed singularities and asymptotics. In particular, $p(X)$ has poles with equal residue $-\Delta/4g$ at the points $X = \pm 1$ and can also have branch-cuts denoted $\mathcal{C}_k$ for $k = 1, \ldots, K$. Its discontinuity across each cut is fixed by the equation,

$$p(X + i\epsilon) + p(X - i\epsilon) = 2\pi n_k \tag{3.19}$$

for all $X \in \mathcal{C}_k$. The integer n_k associated with each cut is directly related to the mode number of a corresponding string oscillator. The quasi-momentum is properly defined as an abelian integral of a meromorphic differential on an appropriate branched covering of the complex X-plane. The behaviour of the quasi-momentum at these branch cuts can be encoded by expressing it in terms of a resolvent $G(X)$ as,

$$p(X) = G(X) - \frac{\Delta}{4g}\left[\frac{1}{X-1} + \frac{1}{X+1}\right] \tag{3.20}$$

where the resolvent is defined in terms of a positive density $i\rho(X)$ which is non-zero along a contour $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \ldots \cup \mathcal{C}_K$ whose connected component are the branch cuts,

$$G(X) = \int_C dY\, \frac{\rho(Y)}{X - Y}. \tag{3.21}$$

From (3.19), we find that the resolvent satisfies the fundamental equation,

$$G(X + i\epsilon) + G(X - i\epsilon) \equiv 2\fint_{\mathcal{C}} \frac{\rho(Y)}{X - Y}\, dy = 2\pi n_k + \frac{\Delta}{2g}\Big[\frac{1}{X-1} + \frac{1}{X+1}\Big]. \tag{3.22}$$

The conserved charges $E = \Delta - J$, Q and worldsheet momentum p of the classical string solution are each determined in terms of the density $\rho(X)$ as,

$$\int_{\mathcal{C}} dX\, \rho(X) = \frac{1}{2g}\,(E + Q)\ , \tag{3.23}$$

$$\int_{\mathcal{C}} dX\, \frac{\rho(X)}{X} = p\,, \tag{3.24}$$

$$\int_{\mathcal{C}} dX\, \frac{\rho(X)}{X^2} = \frac{1}{2g}\,(E - Q)\ . \tag{3.25}$$

In general the allowed configurations of the density $\rho(X)$ are determined by solving the integral equation (3.22). This leads to families of solutions where ρ varies non-trivially along the square root branch cuts of $p(X)$. The system also admits another type of configuration where $\rho(X)$ remains constant along certain contours in the x-plane. This leads instead to logarithmic branch points of the quasi-momentum. The corresponding branch cuts are referred to as "condensate cuts".

In the present case we are interested in the case of large energy $\Delta >> 1$. In this case, the square root branch cuts shrink to zero size and non-trivial configurations are described by condensate cuts alone. The simplest such configuration is a single condensate cut with constant density $i\rho(x) = 1$ and endpoints at $X = X^+$ and $X = X^-$. The corresponding resolvent is,

$$G(X; X^\pm) = -i \int_{X^-}^{X^+} \frac{dY}{X - Y} = \frac{1}{i} \log\left(\frac{X - X^+}{X - X^-}\right) . \tag{3.26}$$

As we explain below, this is the fundamental quantity we need for obtaining the scattering phase for fluctuations around the dyonic giant magnon. Applying the relations (3.23), (3.25) and (3.24), we immediately obtain respectively the formulae for the conserved charges (2.3), (2.4) and (2.2). We can also eliminate the dependence on the endpoints $X^\pm$ in these expressions to obtain the dispersion relation,

$$E = \sqrt{Q^2 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \tag{3.27}$$

This precisely matches the dispersion relation for the Dyonic Giant Magnon (DGM) solution of classical string theory on $\mathbb{R} \times S^3$ [8] and it is natural to identify the condensate cut configuration described above as the KMMZ spectral data corresponding to this classical solution [44 – 46]. In this classical context, the conserved charge Q is a continuous parameter. The original Giant Magnon solution of Hofman and Maldacena [5] is obtained by taking the limit $Q \to 0$ of this more general configuration.

Now let us consider a perturbation around the dyonic giant magnon solution with resolvent (3.26) described above. In our discussion of the dressing method in the previous subsection, the fluctuation corresponded to the introduction of an additional "small" soliton. The corresponding perturbation of the finite gap data is to introduce a single additional pole in the quasi-momentum $p(X)$ [10, 11]. Roughly speaking this can also be thought of as the limiting configuration obtained by shrinking an additional condensate cut corresponding to an additional DGM. To ensure that the new configuration with the additional simple pole remains a solution to the equations of motion, the position $X = r \in \mathbb{R}$ of the pole is not arbitrary, but is determined by the fundamental equation (3.22) which now reads,

$$2G(r; X^\pm) = 2\pi\tilde{n} + \frac{\Delta}{2g}\left[\frac{1}{r-1} + \frac{1}{r+1}\right], \quad \tilde{n} \in \mathbb{Z}. \tag{3.28}$$

The worldsheet momentum associated with the additional pole at $x = r$ is simply that of a corresponding plane wave excitation (2.9) of wavenumber $k(r) = 2r/(r^2 - 1) = 1/(r - 1) + 1/(r + 1)$. As mentioned above the length ℓ, of the corresponding closed string (measured in the worldsheet coordinate x which is normalised to be conjugate to the wavenumber k) is related to the string energy as $\ell = \Delta/2g$. We then obtain the following equation from (3.28),

$$2G(r; X^\pm) + k(r)\ell = 2\pi\tilde{n}, \quad \tilde{n} \in \mathbb{Z}. \tag{3.29}$$

This equation is responsible for quantising the allowed values of the wave-number $k(r)$. One then immediately recognises the first term on the l.h.s. of the above equation as the additional phase-shift acquired by the plane-wave fluctuation as it travels a full period ℓ of the string,

$$\delta_{Z_2}(r;X^{\pm}) = 2G(r;X^{\pm}) = -2i\log\Big(\frac{r-X^+}{r-X^-}\Big). \tag{3.30}$$

This precisely matches the result given in the previous subsection for the phase shift for excitations inside the $SU(2)$ sector (see Eqn (3.14)) up to an additive constant linearly proportional to the DGM momentum p.[8]

3.2.2 Embedding in full $AdS_5 \times S^5$

We will now apply the method described in the previous subsection to the full $AdS_5 \times S^5$ background to recover the phase shifts for the fluctuations of each worldsheet field in the dyonic giant magnon background (See [10, 11, 21] for earlier work). The full superstring theory is described by a sigma model that has coset target space

$$\frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)},$$

and the Virasoro constraint imposed. An element $\mathbf{g} \in SU(2,2|4)$ has the following form[9]

$$\mathbf{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and the coset can be constructed from the existence of an $\mathbb{Z}_4$-automorphism $\Omega \in \mathrm{Aut}(PSU(2,2|4))$ with

$$\Omega(\mathbf{g}) := \begin{pmatrix} EA^TE & -ECE \\ EB^TE & ED^TE \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can then identify $\mathcal{H} = \Omega(\mathcal{H})$, from which one gets $\mathcal{H} \approx Sp(2,2)\times Sp(4)$. This model is classically integrable, and its Lax connection is

$$\hat{J}(X) = H + \frac{X^2+1}{X^2-1}P - \frac{2X}{X^2-1}\Big(\star P - \Lambda\Big) + \sqrt{\frac{X+1}{X-1}}Q^1 + \sqrt{\frac{X-1}{X+1}}Q^2. \tag{3.31}$$

Its flatness condition reproduces the worldsheet equations of motion for the IIB superstring on $AdS_5 \times S^5$.

A convenient parametrisation for the eigenvalues of the monodromy matrix is given as follows,

$$\Big\{e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4} | e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4}\Big\}.$$

[8] Such additive constants can be attributed to the different basis choices between string and gauge theories c.f. [47], and most importantly such ambiguities do not contribute to the calculations of the energy shift and the one-loop correction to the scattering phase.

[9] $PSU(2,2|4)$ does not allow a matrix representation.

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The quasi-momenta $\hat{p}_{1,\ldots,4}$ and $\tilde{p}_{1,\ldots,4}$ will then be meromorphic functions over the spectral curve Γ. They will define the 8-sheets of the Riemann surface that will characterise the solution. These sheets will be connected by a set of cuts $\mathcal{C}_1, \ldots, \mathcal{C}_n$ that define the curve. At these cuts the quasi-momenta can jump by a multiple of 2π,

$$p_i(X+i\epsilon) - p_j(X-i\epsilon) = 2\pi n_{ij}, \quad X \in \mathcal{C}_k^{ij}\,. \tag{3.32}$$

This equation is the generalisation of Eqn (3.19) appearing in the analysis of the previous section.

The monodromy matrix obeys [10, 11] the equation

$$C^{-1}\Omega(X)C = \Omega^{-ST}(1/X), \quad \text{with} \quad C = \begin{pmatrix} E & 0 \\ 0 & -iE \end{pmatrix}.$$

This symmetry of the monodromy matrix translates into the following equations for the quasi-momenta,

$$\tilde{p}_{1,2}(X) = -\tilde{p}_{2,1}(1/X), \tag{3.33}$$

$$\tilde{p}_{3,4}(X) = -\tilde{p}_{4,3}(1/X), \tag{3.34}$$

$$\tilde{p}_{1,2,3,4}(X) = -\tilde{p}_{2,1,4,3}(1/X). \tag{3.35}$$

These will be of ultimate importance in fixing the quasi-momenta on all the sheets.

To determine the spectral curve corresponding to finite gap solution that giving rise to the dyonic giant magnon, we make use of this symmetry to embed the $SU(2)$ sector solution in the full theory,

$$\tilde{p}_2(X) = -\tilde{p}_3(X) = p_{SU(2)}(X) = G(X;X^{\pm}) - \frac{\Delta}{2g}\frac{X}{X^2-1}.$$

From this and the (3.33) above we obtain,

$$\tilde{p}_1(X) = -\tilde{p}_2\Big(1/X\Big) = -G\Big(1/X;X^{\pm}\Big) + \frac{\Delta}{2g}\frac{1/X}{1/X^2-1} = -G\Big(1/X;X^{\pm}\Big) - \frac{\Delta}{2g}\frac{X}{X^2-1}.$$

Likewise from (3.34) we obtain

$$\tilde{p}_4(X) = -\tilde{p}_3\Big(1/X\Big) = p_{SU(2)}\Big(1/X\Big) = G\Big(1/X;X^{\pm}\Big) + \frac{\Delta}{2g}\frac{X}{X^2-1}.$$

Repeating the same procedure we determine the relations between all quasi-momenta and the $\mathfrak{su}(2)$ sub-sector resolvent $G(X)$,

$$\tilde{p}_1(X) = -\tilde{p}_4(X) = -G\Big(1/X;X^{\pm}\Big) - \frac{\Delta}{2g}\frac{X}{X^2-1}, \tag{3.36}$$

$$\tilde{p}_2(X) = -\tilde{p}_3(X) = G(X;X^{\pm}) - \frac{\Delta}{2g}\frac{X}{X^2-1}, \tag{3.37}$$

$$\hat{p}_{1,2}(X) = -\hat{p}_{3,4}(X) = -\frac{\Delta}{2g}\frac{X}{X^2-1}. \tag{3.38}$$

We now apply the same method as before: we introduce a microscopic probe cut (or, more simply, a pole), corresponding to a small fluctuation, which can connect any of the eight-sheets. The connection between the excitations of specific worldsheet fields and cuts connecting particular pairs of sheets of the spectral curve was given in [10]:

$$S^5: \quad (i,j) = \overbrace{(\tilde{1},\tilde{3})}^{Z_3},\ \overbrace{(\tilde{1},\tilde{4})}^{\bar{Z}_2},\ \overbrace{(\tilde{2},\tilde{3})}^{Z_2},\ \overbrace{(\tilde{2},\tilde{4})}^{\bar{Z}_3}, \tag{3.39}$$

$$AdS_5: \quad (i,j) = \overbrace{(\hat{1},\hat{3}),\ (\hat{1},\hat{4}),\ (\hat{2},\hat{3}),\ (\hat{2},\hat{4})}^{Y_2,\bar{Y}_2,Y_3,\bar{Y}_3}, \tag{3.40}$$

$$\text{fermionic}: \quad (i,j) = \overbrace{(\tilde{1},\hat{3}),\ (\hat{1},\tilde{4}),\ (\tilde{2},\hat{3}),\ (\hat{2},\tilde{4})}^{\eta_1,\eta_2,\eta_3,\eta_4},\overbrace{(\hat{1},\tilde{3}),\ (\tilde{1},\hat{4}),\ (\hat{2},\tilde{3}),\ (\tilde{2},\hat{4})}^{\theta_1,\theta_2,\theta_3,\theta_4}. \tag{3.41}$$

So for instance a cut connecting the sheets $\tilde{2}$ and $\tilde{3}$ will be a perturbation inside $S^3 \subset S^5$ associated with the $\mathfrak{su}(2)$ sub-sector, i.e., it will be a fluctuation with a polarisation along Z_2. Applying the KMMZ equation to the probe cut we will then have

$$\tilde{p}_2(r) - \tilde{p}_3(r) = 2\pi n_{23}, \quad n_{23} \in \mathbb{Z},$$

that translates, in the language of the $\mathfrak{su}(2)$ sector as

$$2G(r;X^{\pm}) - k(r)\ell = 2\pi\tilde{n}, \quad \tilde{n} \in \mathbb{Z},$$

which coincides with Eqn (3.29).

Repeating this to all other polarisations, we get for the full S^5 sector

$$\tilde{p}_1(r) - \tilde{p}_3(r) = 2\pi n_{13} \Rightarrow G(r;X^{\pm}) - G(1/r;X^{\pm}) - k(r)\ell = 2\pi n_{13}, \tag{3.42}$$
$$\tilde{p}_1(r) - \tilde{p}_4(r) = 2\pi n_{14} \Rightarrow -2G(1/r;X^{\pm}) - k(r)\ell = 2\pi n_{14}, \tag{3.43}$$
$$\tilde{p}_2(r) - \tilde{p}_3(r) = 2\pi n_{23} \Rightarrow 2G(r;X^{\pm}) - k(r)\ell = 2\pi n_{23}, \tag{3.44}$$
$$\tilde{p}_2(r) - \tilde{p}_4(r) = 2\pi n_{24} \Rightarrow G(r;X^{\pm}) - G(1/r;X^{\pm}) - k(r)\ell = 2\pi n_{24}. \tag{3.45}$$

For the AdS_5 sector, these are trivial as expected:

$$k(r)\ell = 2\pi n_{13} = 2\pi n_{14} = 2\pi n_{23} = 2\pi n_{24}\,.$$

Lastly for the fermions we have,

$$-G(1/r;X^{\pm}) - k(r)\ell = 2\pi n_{\tilde{1}\hat{3}} = 2\pi n_{\tilde{1}\hat{4}} = 2\pi n_{\hat{1}\tilde{4}} = 2\pi n_{\hat{2}\tilde{4}}\,, \tag{3.46}$$
$$G(r;X^{\pm}) - k(r)\ell = 2\pi n_{\tilde{2}\hat{3}} = 2\pi n_{\tilde{2}\hat{4}} = 2\pi n_{\hat{1}\tilde{3}} = 2\pi n_{\hat{2}\tilde{3}}\,. \tag{3.47}$$

where in all of these equations $G(r;X^{\pm})$ is the $SU(2)$ resolvent for a dyonic giant magnon solution,

$$G(r;X^{\pm}) = \frac{1}{i}\log\Big(\frac{r-X^+}{r-X^-}\Big), \tag{3.48}$$

and

$$k(r) = \frac{2r}{r^2-1}.$$

Comparing these equations (3.42)–(3.47) with the periodicity equation (3.29) one can immediately read off the various phase shifts:

- For the $SU(2)$ or S^3 sub-sector:

$$\delta_{Z_2}(r;X^{\pm}) \equiv \delta_{\tilde{2}\tilde{3}}(r;X^{\pm}) = 2G(r;X^{\pm}), \tag{3.49}$$

$$\delta_{\bar{Z}_2}(r;X^{\pm}) \equiv \delta_{\tilde{1}\tilde{4}}(r;X^{\pm}) = -2G(1/r;X^{\pm})\,. \tag{3.50}$$

- For the remaining fluctuations within S^5,

$$\delta_{Z_3}(r;X^{\pm}) \equiv \delta_{\tilde{1}\tilde{3}}(r;X^{\pm}) = G(r;X^{\pm}) - G(1/r;X^{\pm}), \tag{3.51}$$

$$\delta_{\bar{Z}_3}(r;X^{\pm}) \equiv \delta_{\tilde{2}\tilde{4}}(r;X^{\pm}) = G(r,X^{\pm}) - G(1/r;X^{\pm})\,. \tag{3.52}$$

- For AdS_5,

$$\delta_{\hat{1}\hat{3}} = \delta_{\hat{1}\hat{4}} = \delta_{\hat{2}\hat{3}} = \delta_{\hat{2}\hat{4}} = 0 \quad \Leftrightarrow \quad \delta_I(r;X^{\pm}) = 0, \tag{3.53}$$

  for $I \in \mathcal{I}_{AdS_5} \equiv \{Y_2, \bar{Y}_2, Y_3, \bar{Y}_3\}$

- Finally for the eight fermions $\mathcal{I} = \mathcal{I}_\theta \cup \mathcal{I}_\eta$,

$$\begin{gathered} \delta_{\hat{1}\tilde{3}}(r;X^{\pm}) = \delta_{\tilde{1}\hat{4}}(r;X^{\pm}) = \delta_{\hat{2}\tilde{3}}(r;X^{\pm}) = \delta_{\tilde{2}\hat{4}}(r;X^{\pm}) = G(r;X^{\pm}) \\ \Updownarrow \\ \delta_I(r;X^{\pm}) = G(r;X^{\pm}), \quad I \in \mathcal{I}_\theta \equiv \{\theta_i\}_{i=1,\ldots,4}, \end{gathered} \tag{3.54}$$

$$\begin{gathered} \delta_{\tilde{1}\hat{3}}(r;X^{\pm}) = \delta_{\hat{1}\tilde{4}}(r;X^{\pm}) = \delta_{\tilde{2}\hat{3}}(r;X^{\pm}) = \delta_{\hat{2}\tilde{4}}(r;X^{\pm}) = -G(1/r;X^{\pm}) \\ \Updownarrow \\ \delta_I(r;X^{\pm}) = -G(1/r;X^{\pm}), \quad I \in \mathcal{I}_\eta \equiv \{\eta_i\}_{i=1,\ldots,4}. \end{gathered} \tag{3.55}$$

In the GM limit $X^{\pm} \to x^{\pm} \equiv \exp(\pm ip/2)$ these simplify to

$$\begin{cases} \delta_I(r;x^{\pm}) = 0, & I \in \mathcal{I}_{AdS_5}, \\ \delta_I(r;x^{\pm}) = -2i\log\left(\frac{r-x^+}{r-x^-}\right), & I \in \mathcal{I}_{S^5}, \\ \delta_I(r;x^{\pm}) = -i\log\left(\frac{r-x^+}{r-x^-}\right), & I \in \mathcal{I}_{\text{fermions}}. \end{cases} \tag{3.56}$$

The results obtained in this section thus agree (up to a constant in the DGM momentum p) with the results from the dressing method for the S^5 sector - compare (3.49-3.52) with (3.14-3.15) using (3.48). In the GM limit we reproduce also the phase shifts determined for the fermions from their explicit solution (see Eqn (3.18)).

3.3 Phase shift from $\mathfrak{su}(2|2)$ S-matrix

In this subsection, we shall consider yet another way of deriving the classical phase shifts for the worldsheet fields in the Giant Magnon background. We will exploit a relation between the phase shifts and a particular weak-coupling limit of the exact magnon S-matrix. In particular, we will take the exact S-matrix for two magnons and take the Giant Magnon limit for one of the incoming particles and the plane-wave limit for the other. In this case the first magnon will become a semiclassical worldsheet soliton and the second an elementary quantum corresponding to a small fluctuation of the worldsheet fields around the soliton background. In such a limit the phase of the S-matrix goes over to the classical phase shift we seek. By varying the polarisations of the second magnon we can select the phase shift corresponding to each worldsheet field. Of course our ultimate goal is to *test* the exact S-matrix at one-loop order so this may sound like a circular argument. However, the calculation of the classical phase shift discussed in this subsection relies only on the well-tested *tree-level* contribution to the exact S-matrix (the AFS phase) as well as the index structure of the S-matrix which is completely determined by supersymmetry [3]. The calculation here should be considered as a consistency check for the results obtained from the finite-gap solution and the dressing method. One drawback is that we only know the full S-matrix for ordinary magnons and not for their bound states.[10] This means that we can only extract the phase shifts for scattering in the background of a charge-less Giant Magnon and not in the more general case of the Dyonic Giant Magnon described above. On the other hand this approach does not require one to choose the polarisation of the background or the "large magnon", as it was the case in the previous sections, hence the universality of the semiclassical correction $\theta_1(x^\pm, y^\pm)$ is more apparent.

To begin with let us recall the schematic form for the full scattering matrix for the elementary magnons given of all sixteen possible flavors given in [3]

$$s(x,y) = s^0(x,y)\left[\hat{s}(x,y)\otimes \hat{s}'(x,y)\right], \tag{3.57}$$

where the abelian factor $s^0(x,y)$ is given by

$$s^0(x,y) = \frac{x^- - y^+}{x^+ - y^-}\frac{1-1/x^+y^-}{1-1/x^-y^+}\sigma^2(x,y). \tag{3.58}$$

The scattering matrix (3.57) was obtained by demanding its invariance under the residual symmetry algebra $\mathfrak{psu}(2|2)\times\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$, and it has been shown to satisfy both unitarity and Yang-Baxter equation. To recover the $\mathfrak{su}(2)$ magnon scattering matrix in (2.16), one simply has to fix the polarisation of the magnon and isolate the relevant component. Moreover as argued in [3], instead of dealing with all (16^4) components of (3.57), we can treat the two copies of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ independently and only identify their central charges. This greatly reduces the number of the components we need to deal with to $4^4 = 256$ and we only need to consider the $\mathfrak{su}(2|2)$ dynamics scattering matrix $\hat{s}(x,y)$.

Recall that the action of $\mathfrak{su}(2|2)$ dynamic S-matrix $\hat{s}(x_j,x_k)$ on a two excitation state is schematically given by

$$\hat{s}(x_j,x_k)|\ldots\mathcal{X}_j\mathcal{X}'_k\ldots\rangle \to (\text{Coeff.})|\ldots\mathcal{X}''_k\mathcal{X}'''_j\ldots\rangle. \tag{3.59}$$

Here an excitation $\mathcal{X}_j$ with spectral parameters $x_j^\pm$ can be any component of the 2+2 dimensional fundamental representation $\{\phi^1,\phi^2|\psi^1,\psi^2\}$ of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$. Notice that in (3.59), under the action of $\hat{s}(x_j,x_k)$, the momenta/spectral parameters of the two excitations have been swapped and their flavors are also allowed to change. As discussed before, in order to derive the leading semiclassical correction $\theta_1(x,y)$ (2.21) to the classical dressing phase, we should consider the scattering between a fluctuation $\mathbf{Z}$ (or elementary magnon in the plane

[10] To do this, it will be necessary to apply the fusion procedure to the entire $\mathfrak{psu}(2|2)^2\ltimes\mathbb{R}^3$ magnon scattering matrix, following [8].

wave regime) with spectral parameters $z^\pm$ and another arbitrary elementary magnon $\mathbf{X}$ with spectral parameters $x^\pm$. We can begin with the full exact expression for the magnon scattering matrix (3.57) but only keep the lowest order $\theta_0(z,x)$ in the dressing phase, which can be readily written as:

$$\exp(ig\theta_0(z,x)) = \frac{1-1/z^-x^+}{1-1/z^+x^-}\left(\frac{1-1/z^-x^+}{1-1/z^+x^+}\frac{1-1/z^+x^-}{1-1/z^-x^-}\right)^{ig(\zeta-u)} . \tag{3.60}$$

Here we have introduced the "rapidity parameters" ζ and u

$$\zeta = z+\frac{1}{z}, \quad u = x+\frac{1}{x}. \tag{3.61}$$

If we further impose the plane wave limit (2.6) on $z^\pm$, $z^+ \sim z^- = r$, the scalar factor $s^0(r;x)$ (3.58) is then simplified to

$$s^0(r;x) = \frac{r-x^+}{r-x^-}\frac{r-1/x^+}{r-1/x^-}. \tag{3.62}$$

The $\mathfrak{su}(2|2)$ scattering matrix $\hat{s}(r;x)$ also simplifies dramatically in the limit (2.6), using the in the notations in (C.39), the only non-vanishing components are:

$$a(r;x) = e(r;x) = \frac{r-x^-}{r-x^+}\sqrt{\frac{x^-}{x^+}}, \quad c(r;x) = -f(r;x) = -1. \tag{3.63}$$

In fact with appropriate choice of the basis for the incoming excitations, $\hat{s}(r;x)$ can be arranged into diagonal form.

Substituting (3.62) and (3.63) into the full expression (3.57), we can the easily obtain the scattering phase between the fluctuation $\mathbf{Z}$ of different polarisations and the arbitrary magnon $\mathbf{X}$. If $\mathbf{Z}$ belongs to one of the four bosonic scalar fluctuations $(\phi_1\tilde{\phi}_1, \phi_1\tilde{\phi}_2, \phi_2\tilde{\phi}_1, \phi_2\tilde{\phi}_2)$ which are identified with string worldsheet fields $\{Z_2, \bar{Z}_2, Z_3, \bar{Z}_3\}$ up to linear combinations (see for example [48] for more precise identifications), its scattering phase with $\mathbf{X}$ is given by

$$\delta(r;x^\pm) = -i\log\left(\frac{r-x^+}{r-x^-}\right) + i\log\left(\frac{r-1/x^+}{r-1/x^-}\right) + p. \tag{3.64}$$

If $\mathbf{Z}$ belongs to one of the four derivatives fluctuations $(\psi_1\tilde{\psi}_1, \psi_1\tilde{\psi}_2, \psi_2\tilde{\psi}_1, \psi_2\tilde{\psi}_2)$ which can be identified with $\{Y_2, \bar{Y}_2, Y_3, \bar{Y}_3\}$, its scattering phase with $\mathbf{X}$ is given by

$$\delta(r;x^\pm) = i\log\left(\frac{r-x^+}{r-x^-}\right) + i\log\left(\frac{r-1/x^+}{r-1/x^-}\right) . \tag{3.65}$$

Finally, if $\mathbf{Z}$ belongs to one of the eight fermionic fluctuations $(\phi_1\tilde{\psi}_1\ \phi_1\tilde{\psi}_2, \phi_2\tilde{\psi}_1, \phi_2\tilde{\psi}_2, \psi_1\tilde{\phi}_1, \psi_1\tilde{\phi}_2, \psi_2\tilde{\phi}_1, \psi_2\tilde{\phi}_2)$ which can be identified with $\{\theta_1,\ldots,\theta_4;\eta_1,\ldots,\eta_4\}$, its scattering phase with $\mathbf{X}$ given by

$$\delta(r;x^\pm) = i\log\left(\frac{r-1/x^+}{r-1/x^-}\right) + \frac{p}{2}. \tag{3.66}$$

Notice that in deriving (3.64)–(3.66), we have not specify the polarisation of $\mathbf{X}$; the point is that one can sure that because of the diagonal form of the reduced $\mathfrak{su}(2|2)$ scattering matrix, the phase shifts derived here are in fact universal and independent of the polarisation of $\mathbf{X}$.

In general, the expressions (3.64)–(3.66) do not coincide with the exact semiclassical phase-shifts calculated from the finite gap solution and the dressing method. This can be explained by the fact that for example in the string sigma model, the exact phase shift was obtained from scattering with dyonic giant magnon, which in turns correspond to the $\mathfrak{su}(2)$ magnon bound states. Here the approach using $\mathfrak{su}(2|2)$ scattering matrix is only strictly valid for the elementary magnons. To make proper comparison with the exact results from sigma model, one should apply the similar bootstrap method used in [8] to the various components here and construct the bound state scattering matrix. However we do expect the results here to match when one consider $\mathbf{X}$ to be in the giant magnon regime (2.11), the exact expressions for the semiclassical phase shift (3.64), (3.66) and (3.65) reduce in such limit to

$$\delta_I(r;x^\pm) = -2i\log\left(\frac{r-x^+}{r-x^-}\right)+p\,, \qquad I\in\mathcal{I}_{\mathrm{S}^5}\,, \tag{3.67}$$

$$\delta_I(r;x^\pm) = -i\log\left(\frac{r-x^-}{r-x^+}\right)+i\log\left(\frac{r-x^-}{r-x^+}\right)=0\,, \qquad I\in\mathcal{I}_{\mathrm{Ad}S_5}\,, \tag{3.68}$$

$$\delta_I(r;x^\pm) = -i\log\left(\frac{r-x^+}{r-x^-}\right)+\frac{p}{2}\,, \qquad I\in\mathcal{I}_{\mathrm{fermions}}\,. \tag{3.69}$$

Respectively, (3.67), (3.68) and (3.69) should compare with the phase shifts experienced by giant magnon due to the scattering with the fluctuations in S^5, in AdS_5 and the fermionic fluctuations; one clearly observes that the expressions (3.67)–(3.69) precisely match with the results from the finite gap solutions and the dressing method up to linear-momentum dependent terms.

4. The zero energy shift and the one-loop correction to the dressing phase

In this section we collect the scattering phases between magnon and fluctuation calculated from various approaches, and apply the formulae (1.10) and (1.19). In the present context these become (after changing variables from k to r in the integrals),

$$\Delta E(X^\pm) = \frac{1}{2\pi}\sum_{I\in\mathcal{I}}(-1)^{F_I}\int_{-1}^{+1}dr\frac{\partial\delta_I(r;X^\pm)}{\partial r}\sqrt{k(r)^2+1}\,, \tag{4.1}$$

$$2\Delta\Theta(X^\pm,Y^\pm) = \frac{1}{2\pi}\sum_{I\in\mathcal{I}}(-1)^{F_I}\int_{-1}^{+1}dr\frac{\partial\delta_I(r;X^\pm)}{\partial r}\delta_I(r;Y^\pm) \tag{4.2}$$

where $k(r)=2r/(r^2-1)$ and where the sums are over all possible polarisations for the intermediate plane-waves, $\mathcal{I}=\mathcal{I}_{AdS_5}\cup\mathcal{I}_{S^5}\cup\mathcal{I}_{\mathrm{fermions}}$. The two dyonic giant magnon are characterised by the spectral data $X^\pm$, $Y^\pm$. The factor of two on the l.h.s. of (4.2) is related to the normalisation for the dressing phase in (3.57). We will now use these formulae to demonstrate the vanishing one-loop energy shift for the soliton and extract the one loop correction to the dressing phase.

It is simple to demonstrate the vanishing energy-shift using the phase-shifts $\delta_I(r; X^{\pm})$ calculated in section 3. We then only need to show the weighted summation over $\delta_I(r; X^{\pm})$ in (4.1) vanishes up to constant r-independent terms. To perform the calculation, one first notes that the fluctuations with a polarisation along AdS_5 will not suffer a phase shift,

$$\delta_I = 0, \quad I \in \mathcal{I}_{AdS_5}.$$

The weighted summation over the phase-shifts for the scattering of the four transverse bosonic fluctuations in S^5 and the eight fermionic fluctuations becomes,

$$\sum_{I\in\mathcal{I}}(-1)^{F_I}\delta_I(r;X^{\pm}) = \overbrace{2G(r;X^{\pm})}^{Z_2}\underbrace{-2G(1/r,X^{\pm})}_{\bar{Z}_2}+$$
$$+\underbrace{2[G(r;X^{\pm})-G(1/r,X^{\pm})]}_{Z_3,\bar{Z}_3}-[\overbrace{4G(r;X^{\pm})}^{\theta_1,\dots,\theta_4}\underbrace{-4G(1/r,X^{\pm})}_{\eta_1,\dots,\eta_4}]=0\,. \quad (4.3)$$

We then automatically obtain from (4.1) the predicted vanishing of the one-loop energy correction for the magnon and its bound states[11]

$$\Delta E = 0. \quad (4.4)$$

We now move on to the one-loop correction to the soliton S-matrix. We are seeking the equality $\Delta\Theta(X^{\pm}, Y^{\pm}) = \Theta_1(X^{\pm}, Y^{\pm})$, where $\Theta_1(X^{\pm}, Y^{\pm})$ is given as

$$\Theta_1(X^{\pm},Y^{\pm}) = K_1(X^+,Y^+) - K_1(X^+,Y^-) - K_1(X^-,Y^+) - K_1(X^-,Y^-)\,. \quad (4.5)$$

Our strategy here is that, instead of comparing with $\Theta_1(X, Y)$ using the expression for $K_1(X, Y)$ in (2.21), (2.29), we shall consider the derivatives of $\Theta_1(X, Y)$ to avoid the issues of the branch cuts coming from the logarithms. Differentiating with respect to $V = (Y^+ + Y^- + 1/Y^+ + 1/Y^-)/2$ we obtain,

$$\frac{\partial\Theta_1(X^{\pm},Y^{\pm})}{\partial V} = \frac{(F_1(X^+,Y^+)-F_1(X^-,Y^+))}{1-1/(Y^+)^2} + \frac{(F_1(X^-,Y^-)-F_1(X^+,Y^-))}{1-1/(Y^-)^2}\,, \quad (4.6)$$

where,

$$F_1(X,Y) = \frac{\partial K_1(X,Y)}{\partial Y} = \frac{1}{\pi}\left[\frac{1}{Y-X} - \frac{1}{Y-1/X}\right]\log\left(\frac{Y+1}{Y-1}\frac{X-1}{X+1}\right)\,, \quad (4.7)$$

and we have used the identities $\frac{\partial Y^{\pm}}{\partial V} = \frac{1}{1-1/(Y^{\pm})^2}$.

[11] A related calculation appeared in [42]. In particular, it was noted that the range and frequencies of the continuous spectra associated with bosonic and fermionic modes were the same. However, as we have emphasized above, to compute the one-loop correction to the soliton energy it is also necessary to determine the appropriate density of states for each mode. See eg [49] for an example where this point is essential.

We shall therefore evaluate the corresponding derivative of our semiclassical result,

$$2\frac{\partial\Delta\Theta(X^\pm,Y^\pm)}{\partial V}=\frac{1}{2\pi}\sum_{I\in\mathcal{I}}(-1)^{F_I}\int_{-1}^{+1}dr\frac{\partial\delta_I(r;X^\pm)}{\partial r}\frac{\partial\delta_I(r;Y^\pm)}{\partial V}\,, \tag{4.8}$$

using the various scattering phases $\delta_I(r;X^\pm)$ between the fluctuations and the magnon polarized in one of the $S^3\subset S^5$ calculated in the previous sections. Instead of evaluating every terms in the weighted summation of (4.8), again the four fluctuations in AdS_5 give vanishing contributions. Moreover each of the two bosonic fluctuations parallel to the S^3 will give four times of the contribution coming from each of the eight fermionic fluctuations, therefore these contributions again cancel after taking account of the multiplicities and weights. As the result, we only need to consider the contributions coming from the two bosonic fluctuations transverse to the S^3, i.e., $\delta_{Z_3}(r;X^\pm)=\delta_{\bar{Z}_3}(r;X^\pm)=G(r;X^\pm)-G(1/r;X^\pm)$. The relevant derivatives are given by:

$$\frac{\partial\delta_{Z_3}(r;X^\pm)}{\partial r}=i\left[\left(\frac{1}{r-X^+}-\frac{1}{r-1/X^+}\right)-\left(\frac{1}{r-X^-}-\frac{1}{r-1/X^-}\right)\right], \tag{4.9}$$

$$\begin{aligned}\frac{\partial\delta_{Z_3}(r;Y^\pm)}{\partial V}=i\Bigg[&\frac{1}{1-1/(Y^+)^2}\left(\frac{1}{Y^+-r}-\frac{1}{Y^+-1/r}\right)\\&-\frac{1}{1-1/(Y^-)^2}\left(\frac{1}{Y^--r}-\frac{1}{Y^--1/r}\right)\Bigg].\end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8), it should be clear that it can be rearranged into

$$2\frac{\partial\Delta\Theta(X^\pm,Y^\pm)}{\partial V}=2\left[\frac{\tilde{F}(X^+,Y^+)-\tilde{F}(X^-,Y^+)}{1-1/(Y^+)^2}+\frac{\tilde{F}(X^-,Y^-)-\tilde{F}(X^+,Y^-)}{1-1/(Y^-)^2}\right], \tag{4.11}$$

where the function $\tilde{F}(X,Y)$ is given by

$$\begin{aligned}\tilde{F}(X,Y)&=\frac{1}{2\pi}\int_{-1}^{+1}dr\left[\frac{1}{r-X}-\frac{1}{r-1/X}\right]\left[\frac{1}{Y-r}-\frac{1}{Y-1/r}\right]\\&=\frac{1}{\pi}\left[\frac{1}{Y-X}-\frac{1}{Y-1/X}\right]\log\left(\frac{Y+1}{Y-1}\frac{X+1}{X-1}\right).\end{aligned} \tag{4.12}$$

In the second line of (4.12) we have used the integrals (D.1) and (D.2) in the appendix D, and we obtain the exact match between $\tilde{F}(X,Y)$ and $F_1(X,Y)$ in (4.6)!

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A. Derivation for the one-loop energy-shift formula

Here we present a derivation for the semiclassical one-loop energy shift formula (4.2). Let us consider a real scalar field $\varphi(x,t)$ in a 1+1 dimensional field theory which contains a mass parameter m and coupling g, we shall consider the strong coupling limit $g\gg 1$ hence the natural expansion parameter is the inverse coupling $1/g$. Now suppose the theory admits a classical one soliton solution $\varphi(x,t) \equiv \varphi_{cl}(x,t;p)$ where p is the conserved momentum carried by the soliton, such solution should have the asymptotic behaviour:

$$\varphi_{cl}(x,t;p) \sim \exp(-c|x|)\,, \quad |x| \to \infty\,, \tag{A.1}$$

where $c \equiv c(p)$ is the mass of the static soliton at rest. The energy of the soliton $E(p,g)$ should also admit the strong coupling expansion in $1/g$ as

$$E(p) = gE_{cl}(p) + \Delta E(p) + \mathcal{O}(1/g)\,, \tag{A.2}$$

where $E_{cl}(p)$ is the classical energy, whereas $\Delta E(p)$ is the semiclassical one-loop energy shift due to the small quantum fluctuations around the classical soliton background.

To determine $\Delta E(p)$, we first consider the standard small fluctuation operator in the soliton theory given by

$$\hat{H} = \frac{\delta^2 \mathcal{L}(\varphi, \partial\varphi)}{\delta\varphi^2(x,t)}\bigg|_{\varphi=\varphi_{cl}(x,t;p)}\,, \tag{A.3}$$

where $\mathcal{L}(\varphi,\partial\varphi)$ is the Lagrangian of the theory. The semiclassical energy shift $\Delta E(p)$ is then determined by the spectrum of $\hat{H}$; asymptotically, i.e. away from the soliton, $\hat{H}$ should tend to quantum mechanical Hamiltonian describing the propagation of plane wave:

$$\hat{H} \to \Box + m^2 + \mathcal{O}\left(e^{-c|x|}\right)\,, \quad |x| \to \infty\,, \tag{A.4}$$

where $\Box = -\partial_t^2 + \partial_x^2$. Hence if we consider a solution $\psi(x,t)$ to the linearised equation of motion, i.e. it satisfies

$$\hat{H}\psi(x,t;k) = 0\,, \quad \psi(x,t;k) \in \mathbb{C}\,. \tag{A.5}$$

Asymptotically, to be consistent with (A.4), the solution $\psi(x,t)$ should have the following behaviour:

$$\begin{aligned} &\psi(x,t;k) \to \exp(iE(k)t + ikx)\,, && x \to -\infty\,,\\ &\psi(x,t;k) \to \exp(i\delta(k;p) + iE(k)t + ikx)\,, && x \to \infty\,, \end{aligned} \tag{A.6}$$

where k is the wave vector of $\psi(x,t;k)$ and $\epsilon(k)$ is an eigenvalue of the asymptotic Hamiltonian (A.4), so that $E(k) = \sqrt{k^2+m^2}$. As it propagates from $x=-\infty$ to $x=\infty$, the fluctuation $\psi(x,t;k)$ scatters elastically with the classical soliton $\varphi_{cl}(x,t;p)$, the unitarity of $\hat{H}$ demands that such scattering can only introduce an overall phase-shift $\delta(k;p)$ into $\psi(x,t;k)$, $\delta(k;p)$ is called the "scattering phase".[12]

We now would like to derive the one-loop energy shift ΔE of the soliton due to the presence of the fluctuation $\psi(x,t;k)$. Instead of considering an infinite line, we now impose periodic boundary condition on the soliton wave function $\varphi_{cl}(x,t;p)$, i.e.

$$\varphi_{cl}(x,t;p) = \varphi_{cl}(x+L,t;p)\,, \quad L \gg 1\,; \tag{A.7}$$

[12] In our analysis, we exclude the possible formation of bound states, and we assume that there is no reflection, however they are indeed true in the case of our interests.

as the result the fluctuation $\psi(x,t;k)$ also acquires the periodicity:

$$\psi(x,t;k) = \psi(x+L,t;k)\,. \tag{A.8}$$

Comparing (A.8) with the asymptotic condition earlier (A.6), we can deduce that the allowed valued of wave vector k_n must satisfy the condition

$$k_n L = 2\pi n + \delta(k_n;p)\,, \quad n \in \mathbb{Z}\,. \tag{A.9}$$

Typically we expect that for a given wave vector $k = k_n$, there should be an unique solution. We can actually impose similar periodic boundary condition in the time direction on the soliton, that is for some given time period T,

$$\varphi_{cl}(x,t;p) = \varphi_{cl}(x,t+T;p)\,. \tag{A.10}$$

Whereas for the fluctuation $\psi(x,t;k)$, after one period T, it picks up a phase given by

$$\psi(x,t+T;k) = \exp(i\nu(k))\psi(x,t;k)\,, \tag{A.11}$$

where $\nu(k) = E(k)T = \sqrt{k^2+m^2}T$, the phase $\nu(k)$ is called "stability angle" in the literature.

Essentially, the derivation for the one-loop energy-shift boils down to comparing the stability angles in the vacuum (without the presence of soliton) and with the existence of soliton. In the vacuum, we can write down the stability angle:

$$\nu(k_n^{(0)}) = = E(k_n^{(0)})T = \sqrt{\left(k_n^{(0)}\right)^2 + m^2}T\,, \tag{A.12}$$

$$Lk_n^{(0)} = 2\pi n\,, \quad n \in \mathbb{Z}\,. \tag{A.13}$$

Here $k_n^{(0)}$ denotes the wave vector for the plane wave propagating in the vacuum and the equation (A.13) is simply the consequence of the periodicity in x-direction. In the soliton background, we can again write down the stability angle for the fluctuation:

$$\nu(k_n) = \sqrt{k_n^2+m^2}T\,, \tag{A.14}$$

with the wave vector k_n now satisfies the periodic condition (A.9). In [14], the general formula for the one loop energy shift such time-dependent solution is given simply as

$$\begin{aligned}\Delta E_L(p) &= \sum_{n=-\infty}^{+\infty}\left(\frac{\partial\nu(k,T)}{\partial T}\bigg|_{k=k_n} - \frac{\partial\nu(k,T)}{\partial T}\bigg|_{k=k_n^{(0)}}\right)\\ &= \sum_{n=-\infty}^{+\infty}\left(\sqrt{k_n^2+m^2} - \sqrt{\left(k_n^{(0)}\right)^2+m^2}\right)\end{aligned} \tag{A.15}$$

As we take the continuous $L\to\infty$ limit, $k_n = \frac{2\pi n}{L} + \mathcal{O}(1/L)$ for high mode numbers $|n| \sim L$, simple algebra shows that $E(k_n) = E(k_n^{(0)}) + \mathcal{O}(1/L)$. In such limit, the summation over the mode number n goes over to an integral, however we can also equivalently express it as integral over the wave vector k, to do so we need to write down the density of states in the soliton background and in the vacuum defined to be:

$$\frac{\partial n}{\partial k} = \frac{L}{2\pi} + \frac{1}{2\pi}\frac{\partial \delta(k;p)}{\partial k}\,, \qquad \frac{\partial n}{\partial k^{(0)}} = \frac{L}{2\pi}\,. \tag{A.16}$$

Finally we deduce the one-loop energy shift formula (A.15) goes over to

$$\begin{aligned}\Delta E(p) &= \lim_{L\to\infty}\left[\Delta E_L(p)\right] = \int_{-\infty}^{+\infty} dk\left(\frac{\partial n}{\partial k} - \frac{\partial n}{\partial k^{(0)}}\right)\sqrt{k^2+m^2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\partial \delta(k;p)}{\partial k}\sqrt{k^2+m^2}\,. \end{aligned} \tag{A.17}$$

For the case of N_F decoupled real fluctuation fields $\psi_I(x,t;k)$ $I = 1,\ldots,N_F$ (include bosonic and fermionic fields), the generalisation is obvious. Furthermore if they all share the same dispersion relations as it is true for the plane wave magnon we consider in this paper, the formula gets extra simplifications, taking into the account of opposite weighting for the bosons and fermions, we finally derive the one loop energy shift formula (4.2):

$$\Delta E(p) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{+\infty} dk \frac{\partial \delta_I(k;p)}{\partial k}\sqrt{k^2+m^2}\,, \tag{A.18}$$

where $\delta_I(k;p)$ corresponds to the scattering phase between the I-th fluctuation and the soliton.

B. Derivation for the one-loop phase shift formula

In this appendix we present the derivation for the formulae of one-loop corrections to the scattering phase given in the equations (1.18) and (1.19). As in the main text, we begin by considering a two soliton solution with momenta p_1 and p_2 respectively in a $1+1$ dimensional field theory characterised by coupling constant g, this configuration can be described by a scattering wave function $\varphi_{\rm scat}(x,t;x_1^{(0)},x_2^{(0)},p_1,p_2)$. In addition we also impose the periodic boundary condition:

$$x \sim x+L\,, \quad \varphi_{\rm scat}(x,t;x_1^{(0)},x_2^{(0)},p_1,p_2) \sim \varphi_{\rm scat}(x+L,t;x_1^{(0)},x_2^{(0)},p_1,p_2) \quad L \gg 1\,, \tag{B.1}$$

this also implies the energy levels of the two solitons are quantised. As the scattering between the two solitons is elastic, the total energy of the system is given by

$$E(n_1,n_2) \equiv E(p_{n_1},p_{n_2}) = E(p_{n_1}) + E(p_{n_2})\,, \qquad n_1,n_2 \in \mathbb{Z}\,, \tag{B.2}$$

Here n_1 and n_2 are again the mode numbers of the two solitons, $E(p_{n_1})$ and $E(p_{n_2})$ their energies, whereas the quantised soliton momenta p_{n_1} and p_{n_2} are given by

$$p_{n_1}L = 2\pi n_1 + \Theta(p_{n_1},p_{n_2})\,, \tag{B.3}$$

$$p_{n_2}L = 2\pi n_2 - \Theta(p_{n_1},p_{n_2})\,. \tag{B.4}$$

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The function $\Theta(p_{n_1}, p_{n_2})$ is the scattering phase between the two solitons, which in general has strong expansion in $1/g$ as given in (1.16), and our aim here is to derive a formula for $\Delta\Theta(p_{n_1}, p_{n_2})$. Notice that the system also has another natural expansion parameter, namely $1/L$ with $L \gg 1$; essentially the set-up of our derivation for $\Delta\Theta(p_{n_1}, p_{n_2})$ is to consider the appropriate double expansions in both $1/g$ and $1/L$ for the soliton momenta and energies p_{n_i} and $E(p_{n_i})$, $i = 1, 2$, and apply (B.3) and (B.4) to relate and identify the terms associated with $\Delta\Theta(p_{n_1}, p_{n_2})$.

Let us begin by expanding the two soliton momenta in $1/L$ while keeping g fixed, we can then write down:

$$p_{n_i} = p_{n_i}^{(0)} + \frac{1}{L} p_{n_i}^{(1)} + \mathcal{O}(1/L^2)\,, \quad i = 1, 2\,. \tag{B.5}$$

If we also divide both sides of (B.3) and (B.4) and replace the momenta entering $\Theta(p_{n_2}, p_{n_2})$ with (B.5), we can obtain that

$$p_{n_1}^{(0)} = \frac{2\pi n_1}{L} \sim \mathcal{O}(1)\,, \quad p_{n_2}^{(0)} = \frac{2\pi n_2}{L} \sim \mathcal{O}(1)\,, \tag{B.6}$$

at the leading order and here we have assumed that the mode numbers n_i to be large so that n_i/L is kept fixed; at the next leading order in $1/L$ expansion we identify that

$$p_{n_1}^{(1)} = -p_{n_2}^{(1)} = \Theta(p_{n_1}^{(0)}, p_{n_2}^{(0)}) = g\Theta_{cl}(p_{n_1}^{(0)}, p_{n_2}^{(0)}) + \Delta\Theta(p_{n_1}^{(0)}, p_{n_2}^{(0)}) + \mathcal{O}(1/g)\,, \tag{B.7}$$

We can also perform a similar expansion for the total energy of the system, which we shall write it as:

$$E(n_1, n_2) = E^{(0)}(n_1, n_2) + \frac{1}{L} E^{(1)}(n_1, n_2) + \mathcal{O}(1/L^2)\,, \tag{B.8}$$

again using (B.5) we can write down

$$E^{(0)}(n_1, n_2) \;=\; E(p_{n_1}^{(0)}) + E(p_{n_2}^{(0)})\,, \tag{B.9}$$

$$E^{(1)}(n_1, n_2) \;=\; g \frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1} = p_{n_1}^{(0)}} \times p_{n_1}^{(1)} + \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2} = p_{n_2}^{(0)}} \times p_{n_2}^{(1)} \tag{B.10}$$

Having expanded in the $1/L$ for the energy, we can now perform further $1/g$ expansions for (B.9) and (B.10), which are can be written as

$$E^{(0)}(n_1, n_2) \;=\; gE_{cl}^{(0)}(n_1, n_2) + \Delta E^{(0)}(n_1, n_2) + \mathcal{O}(1/g)\,, \tag{B.11}$$

$$E^{(1)}(n_1, n_2) \;=\; gE_{cl}^{(1)}(n_1, n_2) + \Delta E^{(1)}(n_1, n_2) + \mathcal{O}(1/g)\,. \tag{B.12}$$

Using the similar double expansion for the energy of individual soliton, we can rewrite the various quantities in (B.11) and (B.12) as the following:

$$E_{cl}^{(0)}(n_1, n_2) \;=\; E_{cl}^{(0)}(n_1) + E_{cl}^{(0)}(n_2)\,, \tag{B.13}$$

$$\Delta E^{(0)}(n_1, n_2) \;=\; \Delta E^{(0)}(n_1) + \Delta E^{(0)}(n_2)\,, \tag{B.14}$$

$$E_{cl}^{(1)}(n_1, n_2) \;=\; g\left[\frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1} = p_{n_1}^{(0)}} - \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2} = p_{n_2}^{(0)}}\right] \Theta(p_{n_1}^{(0)}, p_{n_2}^{(0)})\,, \tag{B.15}$$

$$\Delta E^{(1)}(n_1, n_2) \;=\; g\left[\frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1} = p_{n_1}^{(0)}} - \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2} = p_{n_2}^{(0)}}\right] \Delta\Theta(p_{n_1}^{(0)}, p_{n_2}^{(0)})\,. \tag{B.16}$$

In (B.16) we have used the relation (B.7) and from (B.16) we conclude that we can in fact extract the one loop correction to the soliton scattering phase $\Delta\Theta(p^{(0)}_{n_1}, p^{(0)}_{n_2})$ from the $1/L$ expansion of the one loop energy $\Delta E(n_1, n_2)$! This useful observation allows us to recycle the idea used in deriving the one-loop energy shift for single soliton, that is to consider a plane wave fluctuation with wave vector k_n in the background of two solitons, and we can denote the total one-loop energy shift to be:

$$\Delta E(n_1, n_2) = \sum_{n=-\infty}^{+\infty} \Delta E_n(n_1, n_2)\,, \qquad n \in \mathbb{Z}\,, \tag{B.17}$$

where n is the mode number for the plane wave fluctuation. We assume the plane wave again scatters elastically with the two solitons, moreover the classical integrability of the system persists here, so that the three body scattering matrix can be factorised into pairwise scatterings. We can therefore, at the classical level, write down the periodicity condition for the new three body system:

$$k_n L = 2\pi n + \delta(k_n, p_{n_1}) + \delta(k_n, p_{n_2})\,, \tag{B.18}$$

$$p_{n_1} L = 2\pi n_1 + g\Theta_{cl}(p_{n_1}, p_{n_2}) - \delta(k_n, p_{n_1})\,, \tag{B.19}$$

$$p_{n_2} L = 2\pi n_2 - g\Theta_{cl}(p_{n_1}, p_{n_2}) - \delta(k_n, p_{n_1})\,, \tag{B.20}$$

where $\delta(k_n, p_{n_1})$ and $\delta(k_n, p_{n_2})$ are the scattering phases between the plane wave and the first and second soliton respectively. The $1/L$ expansion in this system yields the expression for k_n

$$k_n = k^{(0)}_n + \frac{1}{L} k^{(1)}_n + \mathcal{O}(1/L^2)\,, \tag{B.21}$$

and we can use the similar arguments for obtaining (B.6) and (B.7) to deduce in this three body case:

$$k^{(0)}_n = \frac{2\pi n}{L}\,, \qquad p^{(0)}_{n_1} = \frac{2\pi n_1}{L}\,, \qquad p^{(0)}_{n_2} = \frac{2\pi n_2}{L}\,, \tag{B.22}$$

$$k^{(1)}_n = \delta(k^{(0)}_n, p^{(0)}_{n_1}) + \delta(k^{(0)}_n, p^{(0)}_{n_2})\,, \tag{B.23}$$

$$p^{(1)}_{n_1} = g\Theta_{cl}(p_{n_1}, p_{n_2}) - \delta(k^{(0)}_n, p_{n_1})\,, \tag{B.24}$$

$$p^{(1)}_{n_2} = -g\Theta_{cl}(p_{n_1}, p_{n_2}) - \delta(k^{(0)}_n, p_{n_2})\,. \tag{B.25}$$

Here in writing out p_{n_1} and p_{n_2} we have not used $p^{(0)}_{n_1}$ and $p^{(0)}_{n_2}$, the point is that we will eventually take the $L \to \infty$ limit, the distinction between them vanish. However for k_n and $k^{(0)}_n$, as we will sum over all infinite mode numbers $-\infty < n < +\infty$ and we expect the summation to go over the integral in the continuous limit, we should therefore be careful with the difference even in such limit. Finally using above, we can write down the $1/L$

expansion for the total energy $E_n(n_1, n_2)$ of this three body system as

$$E_n(n_1,n_2) = \left[gE_{cl}(p_{n_1}^{(0)}) + gE_{cl}(p_{n_2}^{(0)}) + \sqrt{(k_n^{(0)})^2+m^2}\right] \tag{B.26}$$
$$+\frac{g}{L}\left[\frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1}=p_{n_1}^{(0)}} \times p_{n_1}^{(1)} + \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2}=p_{n_2}^{(0)}} \times p_{n_2}^{(1)}\right]$$
$$+\sqrt{k_n^2+m^2} - \sqrt{(k_n^{(0)})^2+m^2}$$
$$-\frac{g}{L}\left[\frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1}=p_{n_1}^{(0)}} \Big(\delta(k_n,p_{n_1}) - \delta(k_n^{(0)},p_{n_1})\Big)\right.$$
$$\left. + \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2}=p_{n_2}^{(0)}} \Big(\delta(k_n,p_{n_2}) - \delta(k_n^{(0)},p_{n_2})\Big)\right].$$

The one-loop energy shift for the two solitons due to the plane wave of wave vector k_n are contained within the last two lines of (B.26), summing over all mode numbers, the total one-loop energy shift due the plane wave is then given by

$$\Delta E(n_1,n_2) = \sum_{n=-\infty}^{+\infty}\left[\sqrt{k_n^2+m^2} - \sqrt{(k_n^{(0)})^2+m^2}\right] \tag{B.27}$$
$$-\frac{g}{L}\sum_{n=-\infty}^{+\infty}\left[\frac{\partial E_{cl}(p_{n_1})}{\partial p_{n_1}}\bigg|_{p_{n_1}=p_{n_1}^{(0)}} \Big(\delta(k_n,p_{n_1}) - \delta(k_n^{(0)},p_{n_1})\Big)\right.$$
$$\left. + \frac{\partial E_{cl}(p_{n_2})}{\partial p_{n_2}}\bigg|_{p_{n_2}=p_{n_2}^{(0)}} \Big(\delta(k_n,p_{n_2}) - \delta(k_n^{(0)},p_{n_2})\Big)\right].$$

If we compare (B.27) with the $1/L$ expansion of $\Delta E(n_1, n_2)$ (B.17):

$$\Delta E(n_1,n_2) = \Delta E^{(0)}(n_1,n_2) + \frac{1}{L}\Delta E^{(1)}(n_1,n_2) + \mathcal{O}(1/L^2)\,, \tag{B.28}$$

as well as apply the explicit expressions (B.14) and (B.16), we can therefore deduce that in the $L \to \infty$ limit

$$\Delta E(n_1,n_2) = \lim_{L\to\infty}\sum_{n=-\infty}^{+\infty}\left[\sqrt{k_n^2+m^2} - \sqrt{(k_n^{(0)})^2+m^2}\right], \tag{B.29}$$

$$\Delta\Theta(p_1,p_2) = \lim_{L\to\infty}\sum_{n=-\infty}^{+\infty}\left[-\delta(k_n,p_{n_1}) + \delta(k_n^{(0)},p_{n_1})\right], \tag{B.30}$$

$$\Delta\Theta(p_1,p_2) = \lim_{L\to\infty}\sum_{n=-\infty}^{+\infty}\left[\delta(k_n,p_{n_2}) - \delta(k_n^{(0)},p_{n_2})\right]. \tag{B.31}$$

The second and third lines above can be calculated independently and used as a consistency check.

To obtain the integral expressions for (B.29)–(B.31), we can recycle the arguments in section A and write down the density of states in this case

$$\frac{\partial n}{\partial k} = \frac{L}{2\pi} + \frac{1}{2\pi}\frac{\partial\delta(k,p_1)}{\partial k} + \frac{1}{2\pi}\frac{\partial\delta(k,p_2)}{\partial k}\,, \qquad \frac{\partial n}{\partial k^{(0)}} = \frac{L}{2\pi}\,. \tag{B.32}$$

Finally plugging in the expressions in (B.32), we can rewrite (B.29) into

$$\begin{aligned}\Delta E(n_1,n_2) &= \lim_{L\to\infty}\sum_{n=-\infty}^{+\infty}\left[\sqrt{k_n^2+m^2}-\sqrt{(k_n^{(0)})^2+m^2}\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\left(\frac{\partial\delta(k,p_1)}{\partial k}+\frac{\partial\delta(k,p_2)}{\partial k}\right)\sqrt{k^2+m^2}\\ &= \Delta E(p_1)+\Delta E(p_2)\,. \end{aligned}\tag{B.33}$$

In this last line of (B.33) we have used the one-loop energy shift formula for single soliton we derived earlier (A.17). Moreover we can use (B.32) to rewrite

$$\begin{aligned}\Delta\Theta(p_1,p_2) &= \lim_{L\to\infty}\sum_{n=-\infty}^{+\infty}\left[-\delta(k_n,p_{n_1})+\delta(k_n^{(0)},p_{n_1})\right]\\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\left(\frac{\partial\delta(k,p_1)}{\partial k}+\frac{\partial\delta(k,p_2)}{\partial k}\right)\delta(k,p_1)\\ &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\left[\frac{1}{2}\frac{\partial}{\partial k}\left[\delta^2(k,p_1)\right]+\frac{\partial}{\partial k}\left[\delta(k,p_1)\delta(k,p_2)\right]-\frac{\partial\delta(k,p_1)}{\partial k}\delta(k,p_2)\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\frac{\partial\delta(k,p_1)}{\partial k}\delta(k,p_2) \end{aligned}\tag{B.34}$$

In the third line of (B.34), we have discarded the total derivative terms; we can also perform similar calculation for (B.31) and show that it is identical to (B.34). In either case, they are indeed the one loop phase shift for the fluctuation of single flavor given in (1.18). For the generalisation, we can consider plane wave fluctuations of different flavors and both bosonic and fermionic, all of them share the same same dispersion relations, we can at last write down the generalised scattering one-loop scattering phase shift:

$$\Delta\Theta(p_1,p_2) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{+\infty}dk\frac{\partial\delta_I(k,p_1)}{\partial k}\delta_I(k,p_2)\,,\tag{B.35}$$

which was stated and used in the main text (c.f. (1.19)) and this completes our derivation.

C. Calculation details for the dressing method

In this appendix we present the calculations for the phase shifts suffered by a plane-wave fluctuation as it scatters a N-soliton DGM string solution lying inside a S^3 subspace of the S^5 using dressing method. The key equation for deriving the asymptotics of the plane-wave solution $\delta\mathbf{g}_N$ is given by:

$$\delta\mathbf{g}_N\Big|_{x\to\pm\infty} = -2i\sin\Big(\frac{q}{2}\Big)\Big(\mathcal{P}_{N+1}[\tilde{w}]+\mathcal{Q}_{N+1}[\tilde{w}]\Big)\Big|_{\tilde{\eta}=0,x\to\pm\infty}\mathbf{g}_N\Big|_{x\to\pm\infty}\,,\tag{C.1}$$

where $\tilde{w}$ is the polarisation vector of the perturbation, q the perturbation momentum and $\mathbf{g}_N$ the N-soliton background solution. We thus have to determine the asymptotics for both the N-soliton solution and for the projectors $\mathcal{P}_{N+1}|_{\tilde{\eta}=0}$, $\mathcal{Q}_{N+1}|_{\tilde{\eta}=0}$.

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Taking the asymptotic limit $x \to \pm\infty$ simplifies the calculation greatly, since we find that

$$\mathcal{P}_1|_{\pm\infty} \approx \begin{pmatrix} \mathcal{P}_1^{\mathrm{SU}(2)}|_{\pm\infty} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{Q}_1|_{\pm\infty} \approx \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{P}_1^{\mathrm{SU}(2)}|_{\pm\infty} \end{pmatrix}, \tag{C.2}$$

where $\mathcal{P}_1^{\mathrm{SU}(2)}|_{+\infty} \approx \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{P}_1^{\mathrm{SU}(2)}|_{-\infty} \approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are the asymptotic limits of the projector of the SU(2) closed sector and that

$$\mathcal{P}_N|_{\pm\infty} = \mathcal{P}_1|_{\pm\infty}, \quad \mathcal{Q}_N|_{\pm\infty} = \mathcal{Q}_1|_{\pm\infty}, \tag{C.3}$$

when all N-solitons have the same polarisation $w_N = \cdots = w_1 = (i, 1, 0, 0)$.

The N-soliton solution can then be reconstructed from Ψ_0, and written in the following factorised form:

$$\Psi_N(X) = \chi_N(X)\chi_{N-1}(X)\cdots\chi_1(X)\Psi_0(X),$$

with

$$\chi_k(X) = 1 + \frac{X_k - \bar{X}_k}{X - X_k}\mathcal{P}_k[w_k] + \frac{1/\bar{X}_1 - 1/X_k}{X - 1/\bar{X}_k}\mathcal{Q}_k[w_k].$$

In particular we will have

$$\chi_k(X)|_{+\infty} \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{X-X_k^-}{X-X_k^+} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{X-1/X_k^+}{X-1/X_k^-} \end{pmatrix}, \qquad \chi_k(X)|_{-\infty} \approx \begin{pmatrix} \frac{X-X_k^-}{X-X_k^+} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{X-1/X_k^+}{X-1/X_k^-} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.4}$$

and so that

$$\Psi_N(X)\Big|_{+\infty} \approx \mathrm{diag}\left(e^{iZ(X)}, \prod_{k=1}^{N} \frac{X - X_k^-}{X - X_k^+} e^{-iZ(X)}, e^{iZ(X)}, \prod_{k=1}^{N} \frac{X - 1/X_k^+}{X - 1/X_k^-} e^{-iZ(X)} \right), \tag{C.5}$$

$$\Psi_N(X)\Big|_{-\infty} \approx \mathrm{diag}\left(\prod_{k=1}^{N} \frac{X - X_k^-}{X - X_k^+} e^{iZ(X)}, e^{-iZ(X)}, \prod_{k=1}^{N} \frac{X - 1/X_k^+}{X - 1/X_k^-} e^{iZ(X)}, e^{-iZ(X)} \right). \tag{C.6}$$

For real $X = \bar{X} = r$ one gets

$$\Psi_N(r)\Big|_{+\infty} \approx \mathrm{diag}\left(e^{i\frac{v}{2}}, \prod_{k=1}^{N} \frac{r - X_k^-}{r - X_k^+} e^{-i\frac{v}{2}}, e^{i\frac{v}{2}}, \prod_{k=1}^{N} \frac{r - 1/X_k^+}{r - 1/X_k^-} e^{-i\frac{v}{2}} \right), \tag{C.7}$$

$$\Psi_N(r)\Big|_{-\infty} \approx \mathrm{diag}\left(\prod_{k=1}^{N} \frac{r - X_k^-}{r - X_k^+} e^{i\frac{v}{2}}, e^{-i\frac{v}{2}}, \prod_{k=1}^{N} \frac{r - 1/X_k^+}{r - 1/X_k^-} e^{i\frac{v}{2}}, e^{-i\frac{v}{2}} \right), \tag{C.8}$$

where

$$v \equiv Z(X) + \bar{Z}(X) = 2Z(r) = \omega t - kx,$$

with $\omega = \sqrt{k^2+1}$ and $k = 2r/(1-r^2)$. If we have taken $X = \bar{X} = 1/r$, we would get an identical set of expressions but with $k \to -k$ (and with $r \to 1/r$).

If one takes $X = 0$ in (C.5) and (C.6), they reduce to

$$\mathbf{g}_N|_{+\infty} = \mathrm{diag}\Big(e^{it}, e^{-it-iP}, e^{it}, e^{-it-iP}\Big), \tag{C.9}$$

$$\mathbf{g}_N|_{-\infty} = \mathrm{diag}\Big(e^{it-iP}, e^{-it}, e^{it-iP}, e^{-it}\Big), \tag{C.10}$$

where $\sum_{k=1}^{N} p_k = P$ is the total momentum. We can always re-scale $\mathbf{g}_N$ by $e^{i\frac{P}{2}}$ to get a more symmetrical expression,

$$\mathbf{g}_N|_{+\infty} = \mathrm{diag}\Big(e^{it+i\frac{P}{2}}, e^{-it-i\frac{P}{2}}, e^{it+i\frac{P}{2}}, e^{-it-i\frac{P}{2}}\Big), \tag{C.11}$$

$$\mathbf{g}_N|_{-\infty} = \mathrm{diag}\Big(e^{it-i\frac{P}{2}}, e^{-it+i\frac{P}{2}}, e^{it-i\frac{P}{2}}, e^{-it+i\frac{P}{2}}\Big). \tag{C.12}$$

What remains is to determine the asymptotic limits of the η-linearised projectors $\mathcal{P}_{N+1}[\tilde{w}]$ and $\mathcal{Q}_{N+1}[\tilde{w}]$. These involve $\Psi_N(r)|_{\pm\infty}$ and $\Psi_N(1/r)|_{\pm\infty}$ respectively, which can be expressed in terms of the asymptotic limits of $\Psi_0(r)$ and $\Psi_0(1/r)$ by applying the dressing method iteratively, using the simplified expressions (C.2), (C.3) for the lower order projectors that we have found out. Explicitly we have

$$\mathcal{P}_{N+1}[\tilde{w}]\Big|_{\pm\infty} = \Psi_N(r)\Big|_{\pm\infty} W_{\mathcal{P}}[\tilde{w}]\, \bar{\Psi}_N(r)\Big|_{\pm\infty}, \tag{C.13}$$

$$\mathcal{Q}_{N+1}[\tilde{w}]\Big|_{\pm\infty} = \Psi_N(1/r)\Big|_{\pm\infty} W_{\mathcal{Q}}[\tilde{w}]\bar{\Psi}_N(1/r)\Big|_{\pm\infty}. \tag{C.14}$$

where

$$W_{\mathcal{P}}[\tilde{w}] = \frac{\tilde{w}\otimes\tilde{w}^\dagger}{\tilde{w}\cdot\tilde{w}^\dagger} \quad \text{and} \quad W_{\mathcal{Q}}[\tilde{w}] = J\frac{\bar{\tilde{w}}\otimes\tilde{w}^T}{\bar{\tilde{w}}\cdot\tilde{w}^T}J^{-1}.$$

From (C.1), (C.7-C.8) and (C.13-C.14) one can easily determine the phase shifts for a given polarisation $\tilde{w}$. The result is that the phase shifts will always be additive, as expected from the factorisable of the system: *The total phase shift experienced by a plane-wave scattering off a N-soliton background is equal to the sum of the individual phase shifts caused by the scattering between a plane wave and each constituent soliton.*

For the two polarisation types that we are considering, we have

$$W_{\mathcal{P}}^{\parallel} \equiv W_{\mathcal{P}}[\tilde{w}_\parallel] = \frac{1}{2}\begin{pmatrix} 1 & i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_{\mathcal{Q}}^{\parallel} \equiv W_{\mathcal{Q}}[\tilde{w}_\parallel] = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{pmatrix},$$

$$W_{\mathcal{P}}^{\perp} \equiv W_{\mathcal{P}}[\tilde{w}_\perp] = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}, \quad W_{\mathcal{Q}}^{\perp} \equiv W_{\mathcal{Q}}[\tilde{w}_\perp] = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.15}$$

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This will give for $\tilde{w} = \tilde{w}_{\parallel}$,

$$\delta Z_1\Big|_{+\infty} = -i\sin\Big(\frac{p}{2}\Big)e^{iv/2}, \qquad \delta Z_1\Big|_{-\infty} = -i\sin\Big(\frac{p}{2}\Big)e^{-iP}e^{iv/2}, \tag{C.16}$$

$$\delta Z_2\Big|_{+\infty} = \sin\Big(\frac{p}{2}\Big)e^{-iP}e^{iv/2}\prod_{j=1}^{N}\frac{r-X_j^+}{r-X_j^-}, \qquad \delta Z_2\Big|_{-\infty} = \sin\Big(\frac{p}{2}\Big)e^{iv/2}\prod_{j=1}^{N}\frac{r-X_j^-}{r-X_j^+}, \tag{C.17}$$

$$\delta Z_3\Big|_{+\infty} = \delta Z_3\Big|_{-\infty} = 0, \tag{C.18}$$

where $P = \sum_{j=1}^{N} p_j$ is the total dyonic giant magnons momentum. For $\tilde{w} = \tilde{w}_{\perp}$ we get,

$$\delta Z_1\Big|_{+\infty} = -i\sin\Big(\frac{p}{2}\Big)e^{iv/2}, \qquad \delta Z_1\Big|_{-\infty} = -i\sin\Big(\frac{p}{2}\Big)e^{-iP}e^{iv/2}, \tag{C.19}$$

$$\delta Z_2\Big|_{+\infty} = \delta Z_2\Big|_{-\infty} = 0, \tag{C.20}$$

$$\delta Z_3\Big|_{+\infty} = \sin\Big(\frac{p}{2}\Big)e^{iv}\prod_{j=1}^{N}\frac{1/r-X_j^-}{1/r-X_j^+}, \qquad \delta Z_3\Big|_{-\infty} = \sin\Big(\frac{p}{2}\Big)e^{iv}\prod_{j=1}^{N}\frac{r-X_j^+}{r-X_j^-}, \tag{C.21}$$

Here we list the resultant phase shifts constructed from dressing method for the scattering between a plane wave and a general N-soliton configuration lying within a given $S^3 \subset S^5$, parameterised by $|Z_1|^2 + |Z_2|^2 = 1$. For the plane wave perturbations that are parallel to the S^3 subspace, with polarisation vector $\tilde{w}_{\parallel} = (i,1,0,0)^T$, we obtain

$$\delta Z_1, \delta\bar{Z}_1: \quad \delta_1\left(r;\{X_j^{\pm}\}\right) = -\delta_{\bar{1}}\left(1/r,\{X_j^{\pm}\}\right) = P, \tag{C.22}$$

$$\delta Z_2, \delta\bar{Z}_2: \quad \delta_2\left(r;\{X_j^{\pm}\}\right) = -\delta_{\bar{2}}\left(1/r,\{X_j^{\pm}\}\right) = -2i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) - P, \tag{C.23}$$

$$\delta Z_3, \delta\bar{Z}_3: \quad \delta_3\left(r;\{X_j^{\pm}\}\right) = -\delta_{\bar{3}}\left(1/r,\{X_j^{\pm}\}\right) = 0. \tag{C.24}$$

If we take the giant magnon limit on the N-solitons $X_j^{\pm} \to x_j^{\pm} \equiv e^{\pm ip_j/2}$ the expressions above reduce to,

$$\delta Z_1, \delta\bar{Z}_1: \qquad \delta_1\left(r;\{x_j\}\right) = -\delta_{\bar{1}}\left(1/r,\{x_j^{\pm}\}\right) = P, \tag{C.25}$$

$$\delta Z_2, \delta\bar{Z}_2: \qquad \delta_2(r;\{x_j^{\pm}\}) = -\delta_{\bar{2}}(1/r,\{x_j^{\pm}\}) = -2i\sum_{j=1}^{N}\log\left(\frac{r-x_j^+}{r-x^-}\right) - P, \tag{C.26}$$

$$\delta Z_3, \delta\bar{Z}_3: \quad \delta_3\left(r;\{x_j^{\pm}\}\right) = -\delta_{\bar{3}}\left(1/r,\{x_j^{\pm}\}\right) = 0. \tag{C.27}$$

For the perturbations that are transverse to the S^3 but within S^5, with the polarisation vector $\tilde{w}_\perp = (i,0,0,1)^T$, we obtain

$$\delta Z_1, \delta\bar{Z}_1: \quad \delta_1\left(r;\{X_j^\pm\}\right) = -\delta_{\bar{1}}\left(1/r,\{X_j^\pm\}\right) = P, \tag{C.28}$$

$$\delta Z_2, \delta\bar{Z}_2: \quad \delta_2\left(r;\{X_j^\pm\}\right) = -\delta_{\bar{2}}\left(1/r,\{X_j^\pm\}\right) = 0, \tag{C.29}$$

$$\begin{aligned}\delta Z_3, \delta\bar{Z}_3: \quad \delta_3\left(r;\{X_j^\pm\}\right) = -\delta_{\bar{3}}\left(1/r,\{X_j^\pm\}\right) &= -i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) \\ &\quad - i\sum_{j=1}^{N}\log\left(\frac{1/r-X_j^-}{1/r-X_j^+}\right).\end{aligned} \tag{C.30}$$

Notice that $\delta_3(r;\{X_j^\pm\}) = \delta_{\bar{3}}(r;\{X_j^\pm\})$ for this polarisation. In the giant magnon limit these expressions again reduce to,

$$\delta Z_1, \delta\bar{Z}_1: \quad \delta_1\left(r;\{x_j^\pm\}\right) = -\delta_{\bar{1}}\left(1/r;\{x_j^\pm\}\right) = P, \tag{C.31}$$

$$\delta Z_2, \delta\bar{Z}_2: \quad \delta_2\left(r;\{x_j^\pm\}\right) = -\delta_{\bar{2}}\left(1/r;\{x_j^\pm\}\right) = 0, \tag{C.32}$$

$$\delta Z_3, \delta\bar{Z}_3: \quad \delta_3\left(r;\{x_j^\pm\}\right) = -\delta_{\bar{3}}\left(1/r,\{x_j^\pm\}\right) = -2i\sum_{j=1}^{N}\log\left(\frac{r-x^+}{r-x^-}\right) - P. \tag{C.33}$$

For the main calculation of this paper, only the non-trivial phase-shifts will be important. We present them again in a more convenient notation. A perturbation with $\tilde{w} \equiv w_\parallel$ will correspond, as it was said, to a plane-wave travelling in a direction parallel to the direction where the background solitons are moving, i.e, in Z_2 and $\bar{Z}_2$. Only in these directions we will have a non-trivial phase shift from the scattering for this particular polarisation $\tilde{w} = w_\parallel$. Hence we will label these by δ_{Z_2} and $\delta_{\bar{Z}_2}$ to refer to a plane-wave travelling along these directions. In the same fashion, for a perturbation with $\tilde{w} \equiv w_\perp$ the scattering will occur in the perpendicular directions Z_3 and $\bar{Z}_3$ to the moving solitons, and so the phase shifts will be represented by $\delta_{Z_3}, \delta_{\bar{Z}_3}$.

$$\delta_{Z_2}\left(r;\{X_j^\pm\}\right) = -\delta_{\bar{Z}_2}\left(1/r,\{X_j^\pm\}\right) = -2i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) - P, \tag{C.34}$$

$$\delta_{Z_3}\left(r;\{X_j^\pm\}\right) = \delta_{\bar{Z}_3}\left(r;\{X_j^\pm\}\right) = -i\sum_{j=1}^{N}\log\left(\frac{r-X_j^+}{r-X_j^-}\right) - i\sum_{j=1}^{N}\log\left(\frac{1/r-X_j^-}{1/r-X_j^+}\right). \tag{C.35}$$

In the GM limit $X_j^\pm \approx \exp(\pm i p_j/2)$ these take the form,

$$\delta_{Z_2}(r;\{x_j^\pm\}) = -\delta_{\bar{Z}_2}(1/r;\{x_j^\pm\}) = -2i\sum_{j=1}^{N}\log\left(\frac{r-x^+}{r-x^-}\right) - P, \tag{C.36}$$

$$\delta_{Z_3}\left(r;\{x_j^\pm\}\right) = \delta_{\bar{Z}_3}\left(r;\{x_j^\pm\}\right) = -2i\sum_{j=1}^{N}\log\left(\frac{r-x^+}{r-x^-}\right) - P. \tag{C.37}$$

C.1 The explicit $\mathfrak{su}(2|2)$ S-matrix

Here we write out the explicit form for the $\mathfrak{su}(2|2)$ dynamic S-matrix entering in (3.57), following the notations used in [47] (See also [48])

$$
\begin{aligned}
\hat{s}(x,y) = \; & a(x,y)(E_1^1 \otimes E_1^1{}' + E_2^2 \otimes E_2^2{}' + E_1^1 \otimes E_2^2{}' + E_2^2 \otimes E_1^1{}') \\
& + b(x,y)(E_1^1 \otimes E_2^2{}' + E_2^2 \otimes E_1^1{}' - E_1^2 \otimes E_2^1{}' - E_2^1 \otimes E_1^2{}') \\
& + c(x,y)(E_3^3 \otimes E_3^3{}' + E_4^4 \otimes E_4^4{}' + E_3^3 \otimes E_4^4{}' + E_4^4 \otimes E_3^3{}') \\
& + d(x,y)(E_3^3 \otimes E_4^4{}' + E_4^4 \otimes E_3^3{}' - E_4^3 \otimes E_3^4{}' - E_3^4 \otimes E_4^3{}') \\
& + e(x,y)(E_1^1 \otimes E_3^3{}' + E_1^1 \otimes E_4^4{}' + E_2^2 \otimes E_4^4{}' + E_2^2 \otimes E_4^4{}') \\
& + f(x,y)(E_3^3 \otimes E_1^1{}' + E_4^4 \otimes E_1^1{}' + E_3^3 \otimes E_2^2{}' + E_4^4 \otimes E_2^2{}') \\
& + g(x,y)(E_1^4 \otimes E_2^3{}' + E_2^3 \otimes E_1^4{}' - E_2^4 \otimes E_1^3{}' - E_1^3 \otimes E_2^4{}') \\
& + h(x,y)(E_3^2 \otimes E_4^1{}' + E_4^1 \otimes E_3^2{}' - E_4^2 \otimes E_3^1{}' - E_3^1 \otimes E_4^2{}') \\
& + k(x,y)(E_3^1 \otimes E_1^3{}' + E_4^1 \otimes E_1^4{}' + E_2^3 \otimes E_2^3{}' + E_4^2 \otimes E_2^4{}') \\
& + l(x,y)(E_3^1 \otimes E_1^3{}' + E_4^1 \otimes E_1^4{}' + E_3^2 \otimes E_2^3{}' + E_4^2 \otimes E_2^4{}') \,. \qquad \text{(C.38)}
\end{aligned}
$$

The various components in (C.38) for two magnons with spectral parameters $x^{\pm}$ and $y^{\pm}$ are given by

$$
\begin{aligned}
a(x,y) &= \frac{x^+ - y^-}{x^- - y^+} \frac{\eta_y \eta_x}{\tilde{\eta}_y \tilde{\eta}_x} \,, \\
b(x,y) &= \frac{(y^- - y^+)(x^- - x^+)(y^+ + x^+)}{(x^+ - y^-)(y^- x^- - y^+ x^+)} \frac{\eta_x \eta_y}{\tilde{\eta}_x \tilde{\eta}_y} \,, \\
c(x,y) &= -1 \,, \\
d(x,y) &= \frac{(y^- - y^+)(x^- - x^+)(y^+ + x^+)}{(y^- - x^+)(y^- x^- - y^+ x^+)} \,, \\
e(x,y) &= \frac{y^- - x^-}{y^+ - x^-} \frac{\eta_x}{\tilde{\eta}_x} \,, \\
f(x,y) &= \frac{x^+ - y^+}{x^- - y^+} \frac{\eta_y}{\tilde{\eta}_y} \,, \\
g(x,y) &= i \frac{(y^- - y^+)(x^- - x^+)(x^+ - y^+)}{(x^+ - y^-)(y^- x^- - y^+ x^+) \tilde{\eta}_y \tilde{\eta}_x} \,, \\
h(x,y) &= i \frac{y^- x^-}{y^+ x^+} \frac{(y^- - y^+)(x^- - x^+)(x^+ - y^+)}{\eta_y \eta_x (y^- - x^+)(1 - y^- x^-)} \,, \\
k(x,y) &= \frac{x^+ - x^-}{x^- - y^+} \frac{\eta_y}{\tilde{\eta}_x} \,, \\
l(x,y) &= \frac{y^+ - y^-}{x^- - y^+} \frac{\eta_x}{\tilde{\eta}_y} \,. \qquad \text{(C.39)}
\end{aligned}
$$

The functions $\eta_x, \eta_y, \tilde{\eta}_x$ and $\tilde{\eta}_y$ are used to account for the difference between the "gauge/spin-chain" and the "string" basis:

$$\text{Gauge}\ :\ \eta_x = \tilde{\eta}_x = \sqrt{i(x^- - x^+)}\,, \quad \eta_y = \tilde{\eta}_y = \sqrt{i(y^- - y^+)}\,; \tag{C.40}$$

$$\text{String}\ :\ \eta_x = \tilde{\eta}_x\sqrt{\frac{y^+}{y^-}} = \sqrt{i(x^- - x^+)\frac{y^+}{y^-}}\,, \quad \eta_y = \tilde{\eta}_y\sqrt{\frac{x^-}{x^+}} = \sqrt{i(y^- - y^+)\frac{x^-}{x^+}}\,. \tag{C.41}$$

Essentially, if we choose the gauge basis (C.40), the components in (C.39) are the same as the ones in [3]). However as we would like to compare the semiclassical phase shifts with the results obtained from the string sigma model calculations, it is in fact necessary for us to select the string basis (C.41) to obtain the exact matches.

D. Useful integrals for the evaluation of semiclassical phase

Here we list the useful integrals for evaluating the semiclassical phase, using the formula (4.2):

$$\int_{-1}^{+1} dr \frac{1}{r-a}\frac{1}{r-b} = \frac{1}{a-b}\left[\log\left(\frac{a-1}{a+1}\right) - \log\left(\frac{b-1}{b+1}\right)\right], \tag{D.1}$$

$$\int_{-1}^{+} dr \frac{1}{r-a}\frac{1}{b-1/r} = \frac{1}{b-1/a}\log\left(\frac{a-1}{a+1}\right) - \frac{1}{b(ab-1)}\log\left(\frac{1-b}{1+b}\right). \tag{D.2}$$

References

[1] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[2] N. Beisert and M. Staudacher, *Long-range PSU*$(2,2|4)$ *Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[3] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082.

[4] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended* SU(2|2) *symmetry*, *J. Stat. Mech.* (2007) P01017 [nlin.SI/0610017].

[5] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 1309 [hep-th/0604135].

[6] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204].

[7] L. Freyhult and C. Kristjansen, *A universality test of the quantum string Bethe ansatz*, *Phys. Lett.* **B 638** (2006) 258 [hep-th/0604069].

[8] H.-Y. Chen, N. Dorey and K. Okamura, *On the scattering of magnon boundstates*, *JHEP* **11** (2006) 035 [hep-th/0608047].

[9] R. Roiban, *Magnon bound-state scattering in gauge and string theory*, *JHEP* **04** (2007) 048 [hep-th/0608049].

[10] N. Gromov and P. Vieira, *The* $AdS_5 \times S^5$ *superstring quantum spectrum from the algebraic curve*, hep-th/0703191.

[11] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, hep-th/0703266.

[12] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226];
N. Beisert, A.A. Tseytlin and K. Zarembo, *Matching quantum strings to quantum spins: one-loop vs. finite-size corrections*, *Nucl. Phys.* **B 715** (2005) 190 [hep-th/0502173];
R. Hernandez, E. Lopez, A. Perianez and G. Sierra, *Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings*, *JHEP* **06** (2005) 011 [hep-th/0502188];
I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in* SL(2) *sector*, *JHEP* **03** (2005) 013 [hep-th/0501203];
N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett.* **B 629** (2005) 102 [hep-th/0509084].

[13] S. Schafer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [hep-th/0507189];
S. Schafer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett.* **B 639** (2006) 571 [hep-th/0602214];
S. Schafer-Nameki, M. Zamaklar and K. Zarembo, *How accurate is the quantum string Bethe ansatz?*, *JHEP* **12** (2006) 020 [hep-th/0610250].

[14] R.F. Dashen, B. Hasslacher and A. Neveu, *The particle spectrum in model field theories from semiclassical functional integral techniques*, *Phys. Rev.* **D 11** (1975) 3424.

[15] L.D. Faddeev and V.E. Korepin, *Quantum theory of solitons: preliminary version*, *Phys. Rept.* **42** (1978) 1.

[16] R. Rajaraman, *Some nonperturbative semiclassical methods in quantum field theory: a pedagogical review*, *Phys. Rept.* **21** (1975) 227.

[17] R. Jackiw and G. Woo, *Semiclassical scattering of quantized nonlinear waves*, *Phys. Rev.* **D 12** (1975) 1643.

[18] M. Spradlin and A. Volovich, *Dressing the giant magnon*, *JHEP* **10** (2006) 012 [hep-th/0607009].

[19] C. Kalousios, M. Spradlin and A. Volovich, *Dressing the giant magnon. II*, *JHEP* **03** (2007) 020 [hep-th/0611033].

[20] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[21] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[22] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys.* **A 39** (2006) 1311 [hep-th/0604175].

[23] H.-Y. Chen, N. Dorey and K. Okamura, *The asymptotic spectrum of the $N = 4$ super Yang-Mills spin chain*, *JHEP* **03** (2007) 005 [hep-th/0610295].

[24] N. Dorey, D.M. Hofman and J. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D 76** (2007) 025011 [hep-th/0703104].

[25] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[26] J. Maldacena and I. Swanson, *Connecting giant magnons to the pp-wave: an interpolating limit of $AdS_5 \times S^5$*, *Phys. Rev.* **D 76** (2007) 026002 [hep-th/0612079].

[27] H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [hep-th/0605155].

[28] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B 778** (2007) 1 [hep-th/0606126].

[29] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[30] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[31] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044].

[32] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[33] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys.* **B 661** (2003) 19 [*Erratum ibid.* **685** (2004) 405] [hep-ph/0208220].

[34] A.V. Kotikov and L.N. Lipatov, *On the highest transcendentality in $N = 4$ SUSY*, *Nucl. Phys.* **B 769** (2007) 217 [hep-th/0611204].

[35] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* (2006) P11014 [hep-th/0603157].

[36] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[37] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S-matrix*, *Phys. Lett.* **B 639** (2006) 378 [hep-th/0604043].

[38] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[39] R.R. Metsaev and A.A. Tseytlin, *Superstring action in $AdS_5 \times S^5$: Kappa-symmetry light cone gauge*, *Phys. Rev.* **D 63** (2001) 046002 [hep-th/0007036].

[40] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[41] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge Univ. Pr., Cambridge, U.K. (2003).

[42] G. Papathanasiou and M. Spradlin, *Semiclassical quantization of the giant magnon*, arXiv:0704.2389.

[43] J.A. Minahan, *Zero modes for the giant magnon*, *JHEP* **02** (2007) 048 [hep-th/0701005].

[44] J.A. Minahan, A. Tirziu and A.A. Tseytlin, *Infinite spin limit of semiclassical string states*, *JHEP* **08** (2006) 049 [hep-th/0606145].

[45] K. Okamura and R. Suzuki, *A perspective on classical strings from complex sine-gordon solitons*, *Phys. Rev.* **D 75** (2007) 046001 [hep-th/0609026].

[46] B. Vicedo, *Giant magnons and singular curves*, hep-th/0703180.

[47] G. Arutyunov, S. Frolov and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **04** (2007) 002 [hep-th/0612229].

[48] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, *Worldsheet scattering in $AdS_5 \times S^5$*, *JHEP* **03** (2007) 094 [hep-th/0611169].

[49] R.K. Kaul and R. Rajaraman, *Soliton energies in supersymmetric theories*, *Phys. Lett.* **B 131** (1983) 357.